AP Calculus BC
Spring 2014

Name: Key!

Unit 2: Parametrics

Day 1 Homework $\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$ $\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\, dy/dx}{dx/dt}}$

In exercises 1-6 (#3 is omitted), find a) dy/dx and b) d^2y/dx^2 in terms of t.

1. $x = 4\sin t$, $y = 2\cos t$

$\frac{dy}{dx} = -\frac{\tan t}{2}$ $\frac{d^2y}{dx^2} = \frac{-\sec^3 t}{8}$

2. $x = \cos t$, $y = \sqrt{3}\cos t$

$\frac{dy}{dx} = \sqrt{3}$ $\frac{d^2y}{dx^2} = 0$

4. $x = 1/t$, $y = -2 + \ln t$

$\frac{dy}{dx} = -t$ $\frac{d^2y}{dx^2} = t^2$

5. $x = t^2 - 3t$, $y = t^3$

$\frac{dy}{dx} = \frac{3t^2}{2t-3}$ $\frac{d^2y}{dx^2} = \frac{6t^2-18t}{(2t-3)^3}$

6. $x = t^2 + t$, $y = t^2 - t$

$\frac{dy}{dx} = \frac{2t-1}{2t+1}$ $\frac{d^2y}{dx^2} = \frac{4}{(2t+1)^3}$

In exercises 7-10, find the points at which the tangent to the curve is a) horizontal, b) vertical

$dy/dt = 0$ $dx/dt = 0$

7. $x = 2 + \cos t$, $y = -1 + \sin t$

(2,0), (2,-2), (1,-1), (3,-1)

8. $x = \sec t$, $y = \tan t$

none

(1,0), (-1,0)

9. $x = 2 - t$, $y = t^3 - 4t$

none

$\left(2-\sqrt{4/3}, \frac{8}{3\sqrt{3}} - \frac{8}{\sqrt{3}}\right), \left(2+\sqrt{4/3}, \frac{-8}{3\sqrt{3}} + \frac{8}{\sqrt{3}}\right)$

10. $x = -2 + 3\cos t$, $y = 1 + 3\sin t$

(-2,4), (-2,-2), (1,1), (-5,1)

In exercises 11-16 (#15 is omitted), SETUP the integral to find the length of the curve defined by each pair of parametric equations. THEN evaluate the integral using your calculator.

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

11. $x = \cos t$, $y = t + \sin t$, $0 \le t \le \pi$

$L = \int_0^{\pi} \sqrt{2 + 2\cos t}\, dt = 4$

12. $x = \frac{(2t+3)^{3/2}}{3}$, $y = t + \frac{t^2}{2}$, $0 \le t \le 3$

$L = \int_0^3 \sqrt{t^2 + 4t + 4}\, dt = 10.5$

13. $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$, $0 \le t \le 1$

$L = \int_0^1 \sqrt{t^4 + t^2}\, dt \approx 0.60948$

14. $x = 8\cos t + 8t\sin t$, $y = 8\sin t - 8t\cos t$, $0 \le t \le \pi/2$

$L = \int_0^{\frac{\pi}{2}} 8t\, dt = \pi^2$

16. $x = e^t - t^2$, $y = t + e^{-t}$, $-1 \le t \le 2$

$L = \int_{-1}^{2} \sqrt{(e^t - 2t)^2 + (1 - e^{-t})^2}\, dt \approx 4.49691$

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$\boxed{\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}}$

Day 2 Homework

In exercises 1-8, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the (a) component form and (b) magnitude of the vector.

1. $3\mathbf{u}$ $\langle 9, -6 \rangle$ $\sqrt{117}$

2. $-2\mathbf{v}$ $\langle 4, -10 \rangle$ $\sqrt{116}$

3. $\mathbf{u} + \mathbf{v}$ $\langle 1, 3 \rangle$ $\sqrt{10}$

4. $\mathbf{u} - \mathbf{v}$ $\langle 5, -7 \rangle$ $\sqrt{74}$

5. $2\mathbf{u} - 3\mathbf{v}$ $\langle 12, -19 \rangle$ $\sqrt{505}$

6. $-2\mathbf{u} + 5\mathbf{v}$ $\langle -16, 29 \rangle$ $\sqrt{1097}$

7. $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$ $\langle \frac{1}{5}, \frac{14}{5} \rangle$ $\frac{\sqrt{197}}{5}$

8. $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$ $\langle -3, \frac{70}{13} \rangle$ $\frac{\sqrt{6421}}{13}$

In exercises 9-12, find the component form of the vector.

9. The vector $\overrightarrow{PQ}$, where $P = (1,3)$ and $Q = (2,-1)$

$\langle 1, -4 \rangle$

10. The vector $\overrightarrow{OP}$, where O is the origin and P is the midpoint of segment RS, where $R = (2,-1)$ and $S = (-4,3)$

$\langle -1, 1 \rangle$

11. The vector from the point $A = (2,3)$ to the origin

$\langle -2, -3 \rangle$

12. The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where $A = (1,-1)$, $B = (2,0)$, $C = (-1,3)$, and $D = (-2,2)$

$\langle 0, 0 \rangle = \vec{0}$

In exercises 23-26, find the unit vectors (four vectors in all) that are tangent and normal to the curve at the given point.

23. $x = \sqrt{t} + 1, \quad y = t + 1 + 2\sqrt{t}, \quad t = 1$ $\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle, \langle \frac{-1}{\sqrt{17}}, \frac{-4}{\sqrt{17}} \rangle, \langle \frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle, \langle \frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}} \rangle$

24. $x = \ln(t-1), \quad y = t - 1, \quad t = 3$ $\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle, \langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle, \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle, \langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle$

25. $x = 4\cos t, \quad y = 5\sin t, \quad t = \pi/3$ $\langle -\frac{4\sqrt{3}}{\sqrt{73}}, \frac{5}{\sqrt{73}} \rangle, \langle \frac{4\sqrt{3}}{\sqrt{73}}, \frac{-5}{\sqrt{73}} \rangle, \langle \frac{-5}{\sqrt{73}}, \frac{-4\sqrt{3}}{\sqrt{73}} \rangle, \langle \frac{5}{\sqrt{73}}, \frac{4\sqrt{3}}{\sqrt{73}} \rangle$

26. $x = 3\cos t, \quad y = 3\sin t, \quad t = -\pi/4$

$\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle, \langle \frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \rangle,$

$\langle \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle, \langle \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \rangle$

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$\frac{dy}{dx} = \frac{4t^3-6t^2}{2t} = 2t^2-3t; \quad \frac{d^2y}{dx^2} = \frac{4t-3}{2t}\Big|_{t=1} = \frac{4-3}{2} = \frac{1}{2}$

27. If $x = t^2 - 1$ and $y = t^4 - 2t^3$, then, when $t = 1$, $\frac{d^2y}{dx^2}$ is

E

A) 1 B) –1 C) 0 D) 3 E) $\frac{1}{2}$

28. If $x = e^\theta \cos\theta$ and $y = e^\theta \sin\theta$, then, when $\theta = \frac{\pi}{2}$, $\frac{dy}{dx}$ is $\frac{e^\theta \sin\theta + e^\theta \cos\theta}{e^\theta \cos\theta - e^\theta \sin\theta}\Big|_{\theta=\frac{\pi}{2}}$

E

A) 1 B) 0 C) $e^{\frac{\pi}{2}}$ D) nonexistent E) –1 $= \frac{e^{\pi/2}(1+0)}{e^{\pi/2}(0-1)} = \frac{1}{-1} = -1$

29. ***Hint: Use the identity for sin(2t) after taking the first derivative. This makes it MUCH easier!!*** $\sin(2t) = 2\sin t\cos t$

If $x = \cos t$ and $y = \cos 2t$, then, $\frac{d^2y}{dx^2}$ $(\sin t \neq 0)$ is $\frac{dy}{dx} = \frac{-2\sin(2t)}{-\sin(t)} = \frac{4\sin t\cos t}{\sin t} = 4\cos t$

B

A) $4\cos t$ B) 4 C) $\frac{4y}{x}$ D) –4 E) $-4\cot t$

$\frac{d^2y}{dx^2} = \frac{-4\sin t}{-\sin t} = 4$

In each of the questions 30-33, a pair of equations that represents a curve parametrically is given.
Choose the answer that is the derivative $\frac{dy}{dx}$.

30. $x = t - \sin t$ and $y = 1 - \cos t$

A

A) $\frac{\sin t}{1-\cos t}$ B) $\frac{1-\cos t}{\sin t}$ C) $\frac{\sin t}{\cos t - 1}$ D) $\frac{1-x}{y}$ E) $\frac{1-\cos t}{t - \sin t}$

31. $x = \cos^3\theta$ and $y = \sin^3\theta$ $= \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$

D

A) $\tan^3\theta$ B) $-\cot\theta$ C) $\cot\theta$ D) $-\tan\theta$ E) $-\tan^2\theta$

32. $x = 1 - e^{-t}$ and $y = t + e^{-t}$ $= \frac{1-e^{-t}}{e^{-t}} = e^t - 1$

E

A) $\frac{e^{-t}}{1-e^{-t}}$ B) $e^{-t} - 1$ C) $e^t + 1$ D) $e^t - e^{-2t}$ E) $e^t - 1$

33. $x = \frac{1}{1-t}$ and $y = 1 - \ln(1-t)$ $(t < 1)$

C

A) $\frac{1}{1-t}$ B) $t - 1$ C) $\frac{1}{x}$ D) $\frac{(1-t)^2}{t}$ E) $1 + \ln x$

$x = \frac{1}{1-t} \Rightarrow 1-t = \frac{1}{x}$

$\frac{\frac{1}{1-t}}{-(1-t)^{-2}(-1)} = \frac{\frac{1}{1-t}}{\frac{1}{(1-t)^2}} = 1-t$

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Day 3 Homework

In exercises 1-4, express each vector as a linear combination of i and j.

1. the vector $\overrightarrow{PQ}$, where $P=(-1,4)$ and $Q=(5,1)$ $6\vec{i}-3\vec{j}$

2. the vector from the point $P=(3,-4)$ to the origin $-3\vec{i}+4\vec{j}$

3. the vectors (a) $\overrightarrow{AB}+\overrightarrow{CD}$ and (b) $\overrightarrow{AB}-\overrightarrow{CD}$, where $A=(-3,0)$, $B=(0,2)$, $C=(4,0)$, $D=(0,-3)$

(a) $-\vec{i}-\vec{j}$ (b) $7\vec{i}+5\vec{j}$

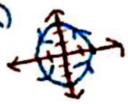
4. the vectors (a) u + v, (b) u − v, (c) 3u, (d) 2u − 3v, where u = 5i − 2j and v = 3i + 4j

(a) $8\vec{i}+2\vec{j}$ (b) $2\vec{i}-6\vec{j}$ (c) $15\vec{i}-6\vec{j}$ (d) $\vec{i}-16\vec{j}$

In exercises 5-8, r(t) is the position vector of a particle in the plane at time t.

(a) Draw the graph of the path of the particle.
(b) Find the velocity and acceleration vectors.
(c) Find the particle's speed and direction of motion and the given value of t.
(d) Write the particle's velocity at that time as the product of its speed and direction.

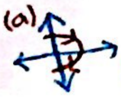
(a)



5. $r(t) = (2\cos t)i + (3\sin t)j$, $t = \pi/2$

(b) $v(t)=(-2\sin t)i+(3\cos t)j$
$a(t)=(-2\cos t)i-(3\sin t)j$
(c) speed $=2$, dir. $=\langle -1,0\rangle$
(d) $(v(t))=2\langle -1,0\rangle$

(a)



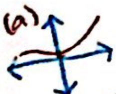
6. $r(t) = (\cos 2t)i + (2\sin t)j$, $t = 0$

(b) $v(t)=(-2\sin(2t))i+(2\cos t)j$
$a(t)=(-4\cos(2t))i-(2\sin t)j$
(c) speed $=2$, dir. $=\langle 0,1\rangle$
(d) $v(t)=2\langle 0,1\rangle$

(a)

7. $r(t) = (\sec t)i + (\tan t)j$, $t = \pi/6$

(a)



8. $r(t) = (2\ln(t + 1))i + (t^2)j$, $t = 1$

In exercises 9 and 10, find an equation for the line that is (a) tangent and (b) normal to the curve r(t) at the point determined by the given value of t.

9. $r(t) = (\sin t)i + (t^2 - \cos t)j$, $t = 0$

10. $r(t) = (2\cos t - 3)i + (3\sin t + 1)j$, $t = \pi/4$

9. (a) through $(0,-1)$ $\to x=0$
(b) through $(0,-1)$ $\to y=-1$

10. (a) $y-\left(\frac{3}{\sqrt{2}}+1\right)=\frac{-3}{2}\left(x-(\sqrt{2}-3)\right)$
(b) $y-\left(\frac{3}{\sqrt{2}}+1\right)=\frac{2}{3}\left(x-(\sqrt{2}-3)\right)$

7. (b) $v(t)=(\sec t\tan t)i+(\sec^2 t)j$
$a(t)=(\sec t\tan^2 t+\sec^3 t)i+(2\sec^2 t\tan t)j$
(c) speed $=\frac{2\sqrt{5}}{3}$, dir. $=\left\langle \frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right\rangle$
(d) $v(t)=\frac{2\sqrt{5}}{3}\langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$

8. (b) $v(t)=\left(\frac{2}{t+1}\right)i+(2t)j$
$a(t)=\left(\frac{-2}{(t+1)^2}\right)i+2j$
(c) speed $=\sqrt{5}$, dir. $=\left\langle \frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right\rangle$
(d) $=\sqrt{5}\left\langle \frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right\rangle$

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Day 4 Homework

In Exercises 11-14, evaluate the integral.

11. $\int_1^2 [(6-6t)\mathbf{i} + 3\sqrt{t}\,\mathbf{j}]\,dt$ $\quad -3\vec{i} + (4\sqrt{2}-2)\vec{j}$

12. $\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1+\cos t)\mathbf{j}]\,dt$ $\quad 0\vec{i} + (\frac{\pi}{2}+\sqrt{2})\vec{j}$

13. $\int [(\sec t\tan t)\mathbf{i} + (\tan t)]\mathbf{j}\,dt$ $\quad (\sec t)\vec{i} + (-\ln|\cos t|)\vec{j} + C$

14. $\int [\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j}]\,dt$ $\quad \ln|t|\vec{i} - \ln|5-t|\vec{j} + C$

In Exercises 15-18, solve the initial value problem for **r** as a vector function of t.

15. $\frac{dr}{dt} = \frac{3}{2}(t+1)^{1/2}\mathbf{i} + e^{-t}\mathbf{j}$, $\mathbf{r}(0) = \mathbf{0}$ $\quad r(t) = ((t+1)^{3/2}-1)\vec{i} + (-e^{-t}+1)\vec{j}$

16. $\frac{dr}{dt} = (t^3+4t)\mathbf{i} + t\,\mathbf{j}$, $\mathbf{r}(0) = \mathbf{i}+\mathbf{j}$ $\quad r(t) = (\frac{t^4}{4}+2t^2+1)\vec{i} + (\frac{t^2}{2}+1)\vec{j}$

17. $\frac{d^2r}{dt^2} = -32\,\mathbf{j}$, $\mathbf{r}(0) = 100\mathbf{i}$, $\left.\frac{dr}{dt}\right|_{t=0} = 8\mathbf{i}+8\mathbf{j}$ $\quad r(t) = (8t+100)\vec{i} + (-16t^2+8t)\vec{j}$

18. $\frac{d^2r}{dt^2} = -\mathbf{i}-\mathbf{j}$, $\mathbf{r}(0) = 10\mathbf{i}+10\mathbf{j}$, $\left.\frac{dr}{dt}\right|_{t=0} = \mathbf{0}$ $\quad r(t) = (\frac{-t^2}{2}+10)\vec{i} + (\frac{-t^2}{2}+10)\vec{j}$

32. The path of a particle $t > 0$ is given by $\mathbf{r}(t) = (t+\frac{2}{t})\mathbf{i} + (3t^2)\mathbf{j}$.
   (a) Find the coordinates of each point on the path where the horizontal component of the velocity of the particle is zero. $\quad (\sqrt{2}+\frac{2}{\sqrt{2}}, 6)$
   (b) Find dy/dx when t = 1. $\quad -6$
   (c) Find d^2y/dx^2 when y = 12. $\quad -24$

34. *Colliding Particles* The paths of two particles for $t \geq 0$ are given by

$$\mathbf{r}_1(t) = (t-3)\mathbf{i} + (t-3)^2\mathbf{j}.$$

$$\mathbf{r}_2(t) = (\tfrac{3t}{2}-4)\mathbf{i} + (\tfrac{3t}{2}-2)\mathbf{j}$$

   (a) Determine the exact time(s) at which the particles collide. $\quad t=2$
   (b) Find the direction of motion of each particle at which time(s) of collision.

$\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle, \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

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