AP Calculus AB

Definitions and Theorems

Contents

Introduction

Success in Calculus requires thorough understanding of key definitions and theorems. It is not sufficient to merely get the general idea of a theorem or to be able to solve only obvious applications. You must understand the significance of every word and symbol with the eye of a lawyer applying a law.

Following is a summary of the top definitions and theorems you will study. There will be additional ones, as well, but these are the foundation upon which everything we study builds.

In addition to popping up on regular quizzes and tests, you will also be given specific definition/theorem assessments to determine how well you have learned them. Following are samples of the types of questions you might see on such an exam. This list is not all-inclusive, but it does cover the most important definitions and theorems.

You are expected to memorize the wording of each definition and theorem, understand the significance of the wording chosen, determine when a definition or theorem can and cannot be applied, and apply the definition or theorem to a specific situation.

Unit 2 – Limits and Continuity

Definition: Vertical Asymptote

The line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad or \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Definition: Horizontal Asymptote

If $\lim_{x\to\infty} f(x) = b$ *or* $\lim_{x\to-\infty} f(x) = b$

Then $y = b$ is a horizontal asymptote of $f(x)$

Definition of Continuity

- **Continuity at an interior point –**

$$\lim_{x \to c} f(x) = f(c)$$

- **Continuity at an endpoint –**

$$\lim_{x \to a^+} f(x) = f(a) \quad or \quad \lim_{x \to b^-} f(x) = f(b)$$

Unit 3 – Derivatives

Definition – Derivative

The derivative of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Provided the limit exists.

Definition (alternate) – Derivative at a Point

The derivative of the function f at the point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x \to a}$$

Provided the limit exists.

Theorem – Differentiability Implies Continuity

If f is differentiable at a, then it is continuous at a.

The converse is NOT true.

Unit 4 – Applications of Derivatives

Definition – Increasing/Decreasing

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f increases on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f decreases on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Theorem
Increasing/Decreasing Functions

Let f be continuous on $[a,b]$ and differentiable on (a,b).

1. If $f' > 0$ at each point of (a,b), then f increases on $[a,b]$.
2. If $f' < 0$ at each point of (a,b), then f decreases on $[a,b]$.

Definition –
Absolute Extreme Values

Let *f* be a function with domain D. Then f(c) is the

(a) absolute maximum value on D if and only if

$$f(x) \le f(c)$$

(b) absolute minimum value on D if and only if

$$f(x) \ge f(c)$$

Definition –
Local Extreme Values

Let *c* be an interior point of the domain of the function *f*. Then f(c) is a

(a) local maximum value at *c* if and only if $f(x) \le f(c)$ for all *x* in some open interval containing *c*.

(b) local minimum value at *c* if and only if $f(x) \ge f(c)$ for all *x* in some open interval containing *c*.

(These definitions consider the entire domain of the function. Considering only a specific interval will be discussed later.)

Theorem – Local Extreme Values

If a function *f* has a local maximum or a local minimum value at an interior point *c* of its domain, and if *f* ' exists at *c*, then

$$f'(c) = 0$$

Definition
Critical Point

A point in the interior of the domain of a function *f* at which *f'* = 0 or does not exist is a **critical point** of *f*.

Extreme Value Theorem

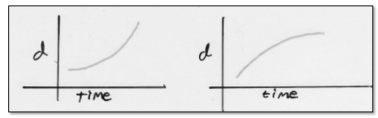
If *f* is continuous on a closed interval, [a,b], then *f* has both a maximum value and a minimum value on that interval.

Definition – Concavity

The graph of a differentiable function $y = f(x)$ is

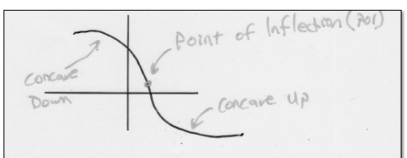
(a) concave up on an open interval I if y' is increasing on I.

(b) concave down on an open interval I if y' is decreasing on I.



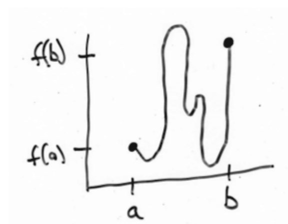
Definition – Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a point of inflection.



Intermediate Value Theorem

If a function is continuous on [a,b] then the function takes on all values between f(a) and f(b).



Mean Value Theorem for Derivatives

If *f* is continuous on [a,b] and differentiable on (a,b)

Then there exists a number, c, in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem

Let *f* be continuous on [a,b] and differentiable on (a,b).

If f(a) = f(b)

Then there is at least one value *c* in (a,b) such that f'(c) = 0

This is a special instance of the Mean Value Theorem

Theorem – Differing Constant

Functions with the same derivative differ by a constant

If $f'(x) = g'(x)$ at each point of an interval I,
then there is a constant C such that $f(x) = g(x) + C$
for all x in I.

Definition – Antiderivative

A function $F(x)$ is an antiderivative of a function $f(x)$
if $F'(x) = f(x)$ for all x in the domain of f.

The process of finding an antiderivative is antidifferentiation.

Unit 5 – Definite Integrals

Definition– The Definite Integral as a Limit of Riemann Sums

Let f be a function on a closed interval [a,b]. For any partition P of [a,b], let the numbers c_k be chosen arbitrarily in the subintervals $[x_{k-1}, x_k]$. If there exists a number I such that

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$$

no matter how P and c_k's are chosen,

Then f is integrable on [a,b] and I is the definite integral of f over [a,b].

The Definite Integral of a Continuous Function on [a,b]

Let f be continous on $[a,b]$, and let $[a.b]$ be partitioned into n subintervals of equal length $\Delta x = (b-a)/n$. Then the definite integral of f over $[a,b]$ is given by

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

where each c_k is chosen arbitrarily in the k^{th} subinterval.

Theorem – The Existence of Definite Integrals

All continuous functions are integrable. That is, if a function f is continous on an interval $[a,b]$, then its definite integral over $[a,b]$ exists.

Definition – Area Under a Curve (as a Definite Integral)

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a,b]$, then the area under the curve $y = f(x)$ from a to b is the integral of f from a to b.

$$A = \int_a^b f(x)dx$$

The Fundamental Theorem of Calculus, Part 1

If f is continuous on [a,b], then the function $F(x) = \int_a^x f(t)\,dt$

(where a is a constant) has a derivative at every point and

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Combining with the chain rule: $\frac{d}{dx}\int_a^u f(t)\,dt = f(u)\frac{du}{dx}$

Fundamental Theorem of Calculus, Part 2 (Evaluation Part)

If f is continuous at every point of $[a,b]$, and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Definition – Average Value

If f is integrable on $[a,b]$, its averable (mean) value on $[a,b]$ is

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Mean Value Theorem for Definite Integrals

If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Sample Test Questions

True or False

The Mean Value Theorem says:

If *f* is continuous on [a,b]

Then there exists a number, c, in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Fill in the blank

The Mean Value Theorem says:

If *f* is _____________ on [a,b] and ______________on (a,b)

Then there exists a number, c, in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Pick the correct term

The Mean Value Theorem says:

If *f* is (*continuous/differentiable*) on [a,b] and (*continuous/ differentiable*) on (a,b)

Then there exists a number, c, in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Fix the mistake

The Mean Value Theorem says:

If *f* is differentiable on [a,b] and continuous on (a,b)

Then there exists a number, c, in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Recite the definition or theorem

The Mean Value Theorem says:

Use the definition or theorem

Given $f(x)=x^2$, find the value, c, guarenteed to exist by the Mean Value Theorem on the interval $[0,2]$.

Can the definition/theorem be used?

Given $f(x)=\frac{1}{x}$, can the Extreme Value Theorem be used to find an absolute maximum and/or minimum on the interval $[0,2]$. Explain why or why not.