BC Calculus

Unit 5 Day 8

Absolute and Conditional Convergence

Ratio and Root Tests

The Alternating Series Test

$$(b)\quad b_n > b_{n+1} \quad for \quad all \quad n$$
$$(c)\quad \lim_{n\to\infty} b_n = 0$$

If part (b) is satisfied
then part (c) tells conclusively if series converges or diverges.

If part (b) is not satisfied then test is inconclusive.

Example

Consider the series $\frac{2}{1}-\frac{1}{1}+\frac{2}{2}-\frac{1}{2}+\frac{2}{3}-\frac{1}{3}+\frac{2}{4}-\frac{1}{4}+\ldots$

The terms are approaching zero but each next term is not necessarily less than the one before it.

The test is INCONCLUSIVE. Some other method of determining convergence is needed.

$$\left(\frac{2}{1}-\frac{1}{1}\right)+\left(\frac{2}{2}-\frac{1}{2}\right)+\left(\frac{2}{3}-\frac{1}{3}\right)+\left(\frac{2}{4}-\frac{1}{4}\right)+\ldots=(1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{4}\right)+\ldots$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}$$

The series is actually equivalent to the harmonic series and diverges.

Alternating Series

If an alternating series is convergent you could then be asked to specify if it is

"Absolutely Convergent"

Or "Conditionally Convergent"

BC Calculus

Absolute Convergence,
Ratio Test, & Root Test

Def. Absolutely Convergent

A series $\sum a_n$ is ABSOLUTELY CONVERGENT if the SERIES $\sum |a_n|$ is convergent.

If a series is absolutely convergent then it is convergent

- Note: We can use ANY test we know to determine absolute convergence!

Example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series (p = 2)

The original series is Absolutely Convergent

The alternating harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

,is convergent by the alternating series test. But is it absolutely convergent?

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$$

This is the harmonic series and is therefore divergent, which means the alternating harmonic series is NOT absolutely convergent

Conditional Convergence

- It is possible for a series to be Convergent, but NOT Absolutely Convergent.
- This is called **CONDITONAL CONVERGENCE**:
  - When the alternating series converges, but the absolute value does not

The possibilities

If $\sum|a_n| = CV$ then $\sum a_n$ also CV and is said to be "absolutely CV" $\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ *vs* $\sum_{n=1}^{\infty}\frac{1}{n^2}$

If $\sum|a_n| = DV$ but $\sum a_n = CV$ then $\sum a_n$ is "conditionally CV" $\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ *vs* $\sum_{n=1}^{\infty}\frac{1}{n}$

If $\sum a_n = DV$ then $\sum|a_n|$ also DV $\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3n}{4n}$ *vs* $\sum_{n=1}^{\infty}\frac{3n}{4n}$

It is not possible for $\sum a_n = DV$ and $\sum|a_n| = CV$

We're almost done.
Just 2 more tests to go.

Is the following series absolutely convergent?

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$$

$$\sum_{n=1}^{\infty}\left|(-1)\frac{n^3}{3^n}\right|=\sum_{n=1}^{\infty}\frac{n^3}{3^n}=\,????$$

We need a test that will work.
Nothing yet in our collection.

The Ratio Test

We are going to compare the ratio between consecutive terms…

(a) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$, *is absolutely convergent (and* $\therefore$ *convergent)*

(b) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, *then the series* $\sum_{n=1}^{\infty} a_n$, *is divergent.*

***L* = 1**

Is inconclusive

Examples:

Test the series for absolute convergence

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n} =$$

Using the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} * \frac{3^n}{n^3} = \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

Now find the limit of this expression.

Examples:

Test the series for absolute convergence

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n} =$$

Using the Ratio Test:

$$\lim_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.

And finally, The ROOT Test

Useful for series like $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$

$\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$ Converges

$\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$ Diverges

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$ Inconclusive

Examples:

1) $$\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n \longrightarrow \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty}\frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

Converges!

2) $$\sum_{n=1}^{\infty}\frac{e^{2n}}{n^n} \longrightarrow \lim_{n\to\infty}\sqrt[n]{\left|\frac{e^{2n}}{n^n}\right|} = \lim_{n\to\infty}\frac{e^2}{n} = 0 < 1$$

Converges!

Packet p.10 (even and odd)
Absolute and Conditional Convergence

2. Abs. CV
3. Abs. CV
4. DV
5. Conditional CV
6. Absolute CV
7. DV
8. Conditional CV
9. Absolute CV
10. DV
11. Absolute CV
12. Absolute CV
13. Absolute CV
14. Absolute CV
15. Absolute CV
16. DV
17. Conditional CV
18.
19. Absolute CV
20. Absolute CV
21. DV
22. Conditional CV
23. Absolute CV
24. Absolute CV
25. Absolute CV
26. DV
27. DV
28. Absolute CV