Announced “Pop” Quiz

- 15 minutes max

Warmup—REVIEW for Test

1. Write the 4th degree Maclaurin polynomial for $f(x) = e^{-x}$. *Save time by using the memorized Maclaurin series for* $g(x) = e^x$

2. *Use your answer to #1 to approximate f(0.25). (Calculator allowed. ☺)*

3. The coefficient of x^4 in the Maclaurin series for $f(x) = e^{-x/2}$ is

$$A. -\frac{1}{24} \quad B. \frac{1}{24} \quad C. \frac{1}{96} \quad D. -\frac{1}{384} \quad E. \frac{1}{384}$$

Warmup—ANSWERS

1. $$f(x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

2. *f(0.25)* ≈ 0.7788

3. E

HW Questions

Remember from yesterday
What the instructions might state . .

Write an n^{th} degree Taylor Polynomial →

Go up to the term of that degree

Write a Taylor Polynomial with n terms →

Write this many, non-zero terms, regardless of degree

Today's Objective

$$f(x) = T_n(x) + R_n(x)$$

Yesterday, we focused on finding the approximating polynomial.

Today we study the remainder (error).

- Use the calculator to analyze the remainder (error)
- Determine the maximum size of the remainder for Alternating Series
- Find the maximum size of the remainder using Taylor's Theorem (aka Lagrange Form)

Two sources of error in a Taylor/Maclaurin approximation

1. **Number of terms of the Taylor polynomial**

   Fewer terms = greater error

2. **How far the *x*-value is from the center point (a)**

   Farther away = greater error

   This means that errors are specific to the x-value you are evaluating. Change the x-value and you need to recalculate the error

<u>Purpose of Taylor Polynomials:</u>

To estimate the value of functions.

AND

it is beneficial to have an idea of how accurate the estimate is.

The following illustrates the relationship between the function, approximate value (partial sum) and the error.

$$f(x) = T_n(x) + R_n(x)$$

Exact Value

Approximate Value

Remainder (error)

How to find the error (remainder):

Calculator Active Method: Graph the absolute value of the difference between the actual function and the Taylor polynomial .

$$f(x) = e^x$$

compared to

$$T(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

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WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
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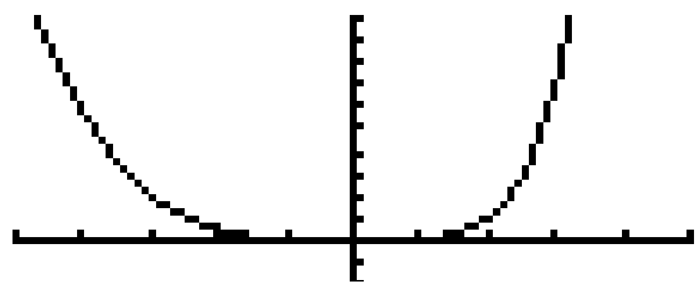
 Plot1 Plot2 Plot3
\Y1=e^(X)
\Y2=1+X+(X²/2)+(
X^3/6)
\Y3=abs(Y1-Y2)
\Y4=
\Y5=
\Y6=
```



To explore—Look at the table of values on the calculator.

Example: **(For alternating series)**

The <u>maximum error</u> of T_n is the a_{n+1} term.

Ex) Find the error of a third degree Maclaurin polynomial that represents $f(x) = e^{-3x}$ for $x = 0.2$

$$T\ \ x\ \ = 1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3$$

This is an alternating series so the <u>maximum error</u> (a.k.a "error bound") will be the absolute value of the n+1 term

$$\left|\frac{81}{4!}(x)^4\right|$$

continued

$$f(x) = 1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3$$

$$R(x) \le \left|\frac{81}{4!}(x)^4\right|$$

$$f(.2) \approx 1 - 3(.2) + \frac{9}{2!}(.2)^2 - \frac{27}{3!}(.2)^3 \approx 0.544$$

$$R(.2) = \left|\frac{81}{4!}(.2)^4\right| = |.0054|$$

This is the maximum error

This means that the following inequality should be true.

$$.544 - .0054 < e^{(-3(.2))} < .544 + .0054$$

In fact, $e^{-.6} = .5488116361$

Example--Practice

Estimate cos(0.2) using a 4^{th} degree Maclaurin Polynomial and find the error bound

Example--Solution

Estimate cos(0.2) using a 4th degree Maclaurin Polynomial and find the error bound

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\cos(0.2) \approx 0.9801$$

The error bound will be the absolute value of the next term evaluated at x=0.2

Absolute value of the next term would be $\left|\frac{-x^6}{6!}\right| = \frac{x^6}{6!}$.

So, the error bound is $\frac{(0.2)^6}{6!} = 8.\bar{8}x10^{-8}$

You Try
Example—Practice #2

Estimate ln(1.3) using a 4th degree Maclaurin Polynomial and find the error bound.

Remember: (Packet p. 5)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Example—Practice #2 (Solution)

Estimate ln(1.3) using a 4[th] degree Maclaurin Polynomial and find the error bound.

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$\ln(1.3) = \ln(1+.3) = .3 - \frac{.3^2}{2} + \frac{.3^3}{3} - \frac{.3^4}{4} \approx 0.2620$$

Absolute value of the next term is: $\dfrac{x^5}{5}$

Error Bound: $\dfrac{.3^5}{5} \approx 4.86 \text{ x } 10^{-4}$

But our series are not always alternating . . . .

Taylor's Theorem w/Remainder

If a function and its (n+1) derivatives are continuous on $|x-a|<r$ then for all x in this interval

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \ldots + f^n(a)\frac{(x-a)^n}{n!} + R_n(x) =$$

$$= T_n(x) + R_n(x)$$

Where

Lagrange Remainder

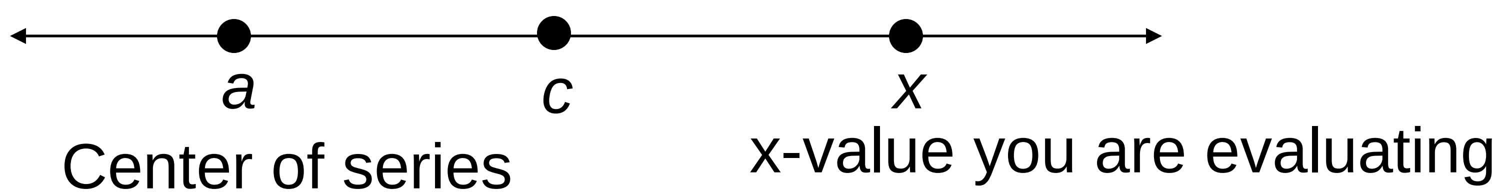
$$R_n(x) = f^{n+1}(c)\frac{(x-a)^{n+1}}{n+1\ !}$$

Taylor's Theorem w/Remainder

Lagrange Remainder

$$R_n(x) = f^{\,n+1}(c)\,\frac{(x-a)^{n+1}}{n+1\;!}$$

This expression for the remainder is the next term in the series at some value "c". The value of "c" is between "a" and some given value of x.



More About Lagrange Remainder

Typically we are most interested in the upper bound of the remainder. This is referred to as the *Lagrange error bound*.

And is found by determining the max of

$$\left| f^{n+1}(c)\frac{(x-a)^{n+1}}{(n+1)!} \right|$$

which will mean

$$R_n < \max\left| f^{n+1}(c)\frac{(x-a)^{n+1}}{(n+1)!} \right|$$

More About Lagrange Remainder

$$R_n < \max\left| f^{n+1}(c)\frac{(x-a)^{n+1}}{(n+1)!} \right|$$

Two requirements:

1. Determine the maximum value of $f^{n+1}(x)$
   a. Using the f^{n+2} derivative

   OR

   b. Knowing the behavior of $f^{n+1}(x)$
2. Maximizing $(x-a)^{n+1}$
   a. If given a specific x-value this is "easy".

Example 1: Type I—Given a specific x-value.

$$T(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2(2!)}x^2 + \frac{\sqrt{2}}{2(3!)}x^3$$ is the 3rd degree Maclaurin polynomial of $f(x) = \cos\left(x + \frac{\pi}{4}\right)$

And $f^4(c) = \cos\left(c + \frac{\pi}{4}\right)$

Determine the error bound when estimating $f(0.1)$

Not Alternating

Therefore, $R_3 \leq \max\left|f^4(c)\frac{x^4}{4!}\right|$ where

x=0.1 and $0 \leq c \leq 0.1$

The goal---Pick a value for c that would maximize $f^4(c) = \cos\left(c + \frac{\pi}{4}\right)$

$$f^{4}(c) = \cos\left(c + \frac{\pi}{4}\right)$$

To determine when this reaches a maximum we need to find the derivative again.

continued

Finding the maximum of $f^4(c) = \cos\left(c + \frac{\pi}{4}\right)$:

$$f^5(c) = -\sin\left(c + \frac{\pi}{4}\right)$$

Which tells us $f^4(c) = \cos\left(c + \frac{\pi}{4}\right)$ is always decreasing on the interval [0, 0.1].

So the maximum would have occurred when c=0 (the starting value of our interval)

Conclusion, the error bound will be . . .

$$R_3 < \max\left|f^4(c)\frac{x^4}{4!}\right|$$

$$= \cos\left(0 + \frac{\pi}{4}\right)\frac{(.1)^4}{4!}$$

$$= \frac{\sqrt{2}}{2}\frac{(.1)^4}{4!}$$

$$= \frac{\sqrt{2}}{2 \cdot 4! \cdot 10^4}$$

Example 2: Still for a given x-value but with less work for determining the maximum of the next derivative.

Use the 3rd degree Maclaurin polynomial $T(x) = x - \frac{x^3}{3!}$,

which is the representation of $f(x) = \sin x$, to find the following:

1. Approximate sin(0.1)

$$\sin(0.1) = 0.1 - \frac{0.1^{\;3}}{3!} = 0.099833$$

2. Determine the level of accuracy of the approximation.

→

Before we look at the level of accuracy let's make sure everybody is following what we are doing. Remember . . .

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$

$$T(x)=x-\frac{x^{3}}{3!}$$

n	$f^{n}(x)$	$f^{n}(a)=f^{n}(0)$
0	$f(x)=\sin x$	$f(0)=0$
1	$f'(x)=\cos x$	$f'(0)=1$
2	$f''(x)=-\sin x$	$f''(0)=0$
3	$f'''(x)=-\cos x$	$f'''(0)=-1$
4	$f^{4}(x)=\sin x$	$f^{4}(0)=0$

→

$$T(x) = x - \frac{x^3}{3!} \qquad f(x) = \sin x$$

n	$f^n(x)$	$f^n(a) = f^n(0)$
0	$f(x) = \sin x$	$f(0) = 0$
1	$f'(x) = \cos x$	$f'(0) = 1$
2	$f''(x) = -\sin x$	$f''(0) = 0$
3	$f'''(x) = -\cos x$	$f'''(0) = -1$
4	$f^4(x) = \sin x$	$f^4(0) = 0$

$$R_n < \max \left| f^{n+1}(c) \frac{(x-a)^{n+1}}{(n+1)!} \right|$$

The maximum value for

$$f^4(x) = \sin x$$

on any interval is 1.

$$\therefore \quad R \; 0.1 \; \le \max \left| f^4(c) \frac{0.1-0 \;\, ^4}{4!} \right| \le 1 \frac{0.1 \;\, ^4}{4!} \approx .0000042$$

→

$$R(0.1) \le \frac{1}{4!}\ 0.1^{4} \approx .0000042$$

$$\sin(0.1) = 0.1 - \frac{0.1^{3}}{3!} = 0.099833$$

$$\sin(0.1) = 0.099833 \pm 0.000004$$

$$0.099829 < \sin(.1) < 0.99837$$

In fact, sin(0.1)=___________

Example 1: Type II—Given an interval of x-values. Similar to your HW problems.

Two requirements:

1. Determine the maximum value of $f^{n+1}(x)$
   a. Using the f^{n+2} derivative

   OR

   b. Knowing the behavior of $f^{n+1}(x)$
2. Maximizing $(x-a)^{n+1}$
   a. If given a specific x-value this is "easy".

This time we will have to be concerned with both requirements.