BC Calculus

Day 6

Taylor & Maclaurin Polynomials

WARMUP

$$g(x)=\frac{e^x-1}{x^2}$$

Find the 1st three terms of a series for g(x) and the nth term.

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e^x-1=\qquad x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$\frac{e^x-1}{x^2}=\frac{x}{x^2}+\frac{x^2}{x^2 2!}+\frac{x^3}{x^2 3!}+\cdots$$

$$=x^{-1}+\frac{1}{2!}+\frac{x}{3!}+\cdots+\frac{x^{n-1}}{(n+1)!}+\cdots$$

Use a Maclaurin series derived in this section to find a Maclaurin series for the following . . . .

Find the Maclaurin series for $\dfrac{(1+\cos 2x)}{2}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad \text{(from previous slide)}$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots$$

$$1 + \cos 2x = 2 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots$$

$$\frac{1+\cos 2x}{2} = \frac{2}{2} - \frac{(2x)^2}{2!2} + \frac{(2x)^4}{4!2} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!2} + \cdots$$

One more example: Packet pg. 2, #11

HW Questions

BC Calculus

Day 6

Taylor & Maclaurin

Polynomials

Did you Memorize These?
(In your packet, page 5.)

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{all real \#s}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \left(-1\right)^n \frac{x^{2n+1}}{\left(2n+1\right)!} + \cdots \qquad \text{all real \#s}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \left(-1\right)^n \frac{x^{2n}}{\left(2n\right)!} + \cdots \qquad \text{all real \#s}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad -1 < x \le 1$$

BUT why memorize these? Saves TIME!! Put this in your NOTES (example)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Find the Maclaurin series for $f(x) = e^{-3x}$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^n}{n!}$$

A neat little problem--NOTES

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = ?$$

This resembles our series for e^x

$$= \sum_{n=0}^{\infty} \frac{3/5 \;^{n}}{n!}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = e^{3/5}$$

Series Converges to this value

But…

How useful is an infinite series if we still cannot determine what it converges to?

We can estimate to whatever accuracy we want using partial sums!

Partial Sums of Taylor Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$T_0 \quad x \quad = 1$$

$$T_1 \quad x \quad = 1 + x$$

$$T_2 \quad x \quad = 1 + x + \frac{x^2}{2!}$$

$$T_3 \quad x \quad = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$T_4 \quad x \quad = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

The more terms we add,

The more accurate we get.

Taylor Polynomial

Notice there is no +. . .

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

T_n is called the

n^{th}-degree ***Taylor polynomial*** of *f* at *a*

$$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

4^{th} degree Taylor Polynomial centered at 0 (i.e. Maclaurin) polynomial of e^x

Key Distinction between Taylor **Series** and Taylor **Polynomial**

Taylor **Series** is the infinite series

Taylor **Polynomial** is a partial sum of a Taylor Series

What the instructions might state . .

Write an n^{th} degree → Go up to the term of that degree

Write n terms → Write this many, non-zero terms, regardless of degree

Taylor Series centered at 0

Find the 5th degree Maclaurin polynomial for *f(x)=*sin(*x*)

n	Derivative	Centered at *a=0*
0	$f(x) = \sin x$	$= 0$
1	$f'(x) = \cos x$	$= 1$
2	$f''(x) = -\sin x$	$= 0$
3	$f'''(x) = -\cos x$	$= -1$
4	$f^{4}(x) = \sin x$	$= 0$
5	$f^{5}(x) = \cos x$	$= 1$

Find the 5th degree Maclaurin polynomial for *f(x)*=sin(*x*) at *a* = 0

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$

Now substituting

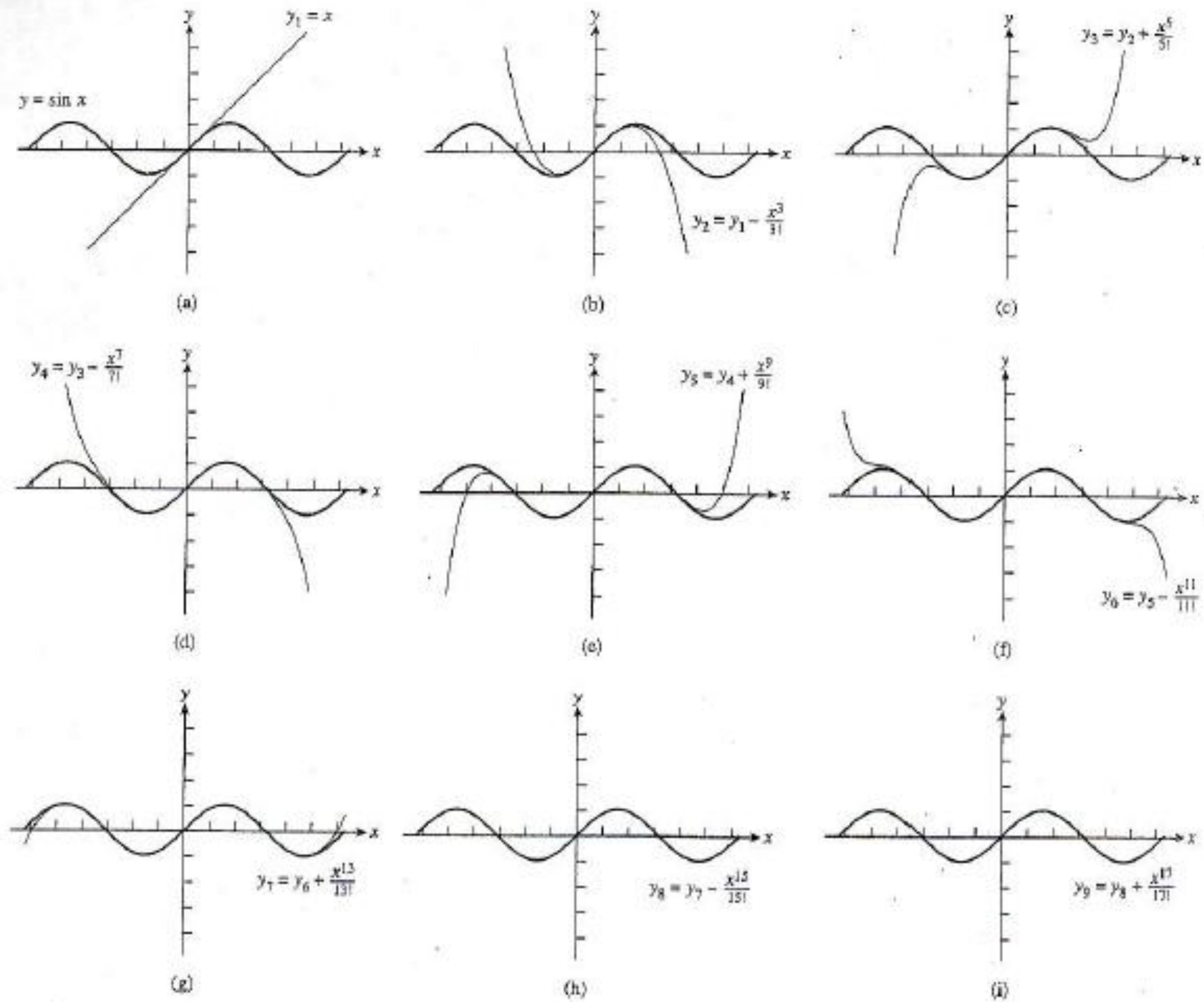
$$\sin x = 0 + x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Instructions state find the 5th degree polynomial, so we stop here.

Why do we care?

The more terms you add to the partial sum, the closer the series fits the function.


y
x
y = sin x
$y_1 = x$
(a)
$y_2 = y_1 - \frac{x^3}{3!}$
(b)
$y_3 = y_2 + \frac{x^5}{5!}$
(c)
$y_4 = y_3 - \frac{x^7}{7!}$
(d)
$y_5 = y_4 + \frac{x^9}{9!}$
(e)
$y_6 = y_5 - \frac{x^{11}}{11!}$
(f)
$y_7 = y_6 + \frac{x^{13}}{13!}$
(g)
$y_8 = y_7 - \frac{x^{15}}{15!}$
(h)
$y_9 = y_8 + \frac{x^{17}}{17!}$
(i)

If we needed the complete series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^{n}$$

Figure out the structure.

Notice the +. . .

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \left(-1\right)^{???} \frac{x^{???}}{(???)!} + \cdots$$

n=0 n=1 n=2

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \left(-1\right)^{n} \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Did you Memorize These??

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{all real \#s}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{all real \#s}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad \text{all real \#s}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad -1 < x \le 1$$

A little REVIEW--Find the Interval of Convergence

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \;\; -1 \;^{n} \frac{x^{2n+1}}{2n+1 \;\; !} + \cdots$$

$$\lim_{n\to\infty}\left| \frac{x^{2(n+1)+1}}{2(n+1)+1 \;\; !} \cdot \frac{2n+1 \;\; !}{x^{2n+1}} \right| = \lim_{n\to\infty}\left| \frac{x^{2n+3}}{2n+3 \;\; !} \cdot \frac{2n+1 \;\; !}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty}\left| \frac{x^2}{2n+2 \quad 2n+3} \right| = 0 \;\; < 1$$

Therefore CV for all real #s

DIFFERENT type of EXAMPLE PROBLEM: Given the following information about a function and its first three derivatives, write the 3rd degree Taylor Polynomial centered at *x = 2*.

$$f(2) = 4$$
$$f'(2) = 3$$
$$f''(2) = 5$$
$$f'''(2) = 7$$

We do not know the function, but we do know its derivative values.

$$T_3(x) = 4 + 3(x-2) + \frac{5}{2}(x-2)^2 + \frac{7}{3!}(x-2)^3$$

$$T'_3(x) = 3 + 5(x-2) + \frac{7}{2}(x-2)^2$$