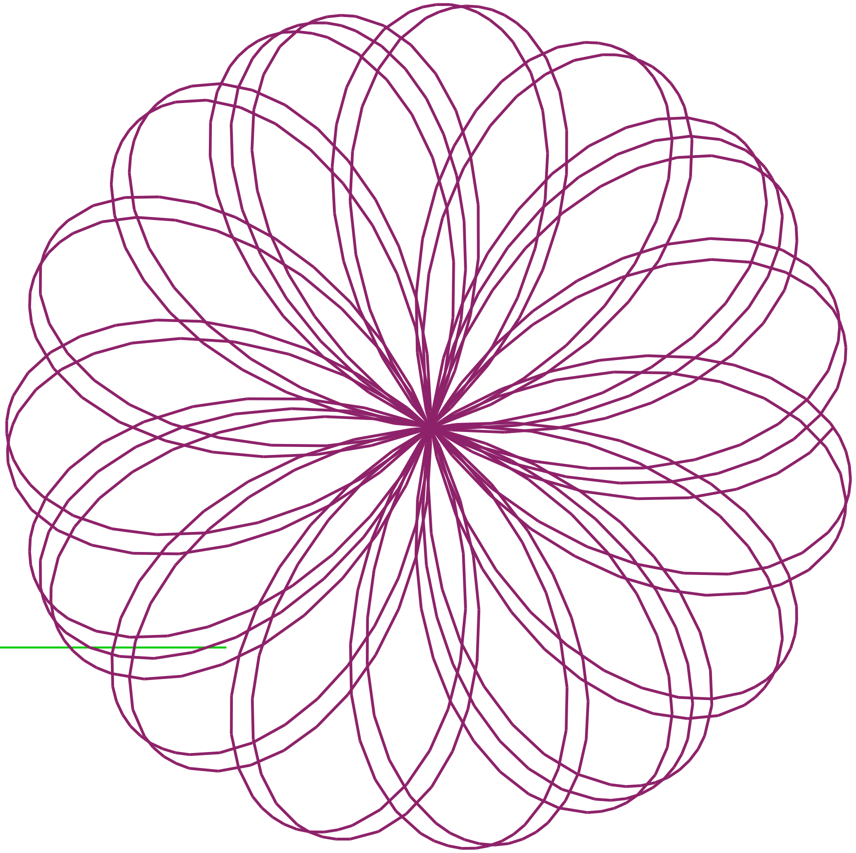
BC Calculus

Unit 1 Day 5



$$r = 2\sin\ 2.15\theta$$

$$0 \le \theta \le 16\pi$$

Warmup

1. The area of the region bounded by $y = e^{2x}$, the y-axis, the x-axis, and the line $x = 2$ is equal to:

$(A)\ \frac{e^4}{2} - e$ $(B)\ \frac{e^4}{2} - 1$ $(C)\ \frac{e^4}{2} - \frac{1}{2}$

$(D)\ 2e^4 - e$ $(E)\ 2e^4 - 2$

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(C) $\dfrac{e^4}{2} - \dfrac{1}{2}$

(D) $2e^4 - e$

(E) $2e^4 - 2$

Today’s Agenda . . . .

- Address questions on “*self-study*” of Length of Polar Curves
- HW Questions
- New topic Area BOUNDED by Polar Curve

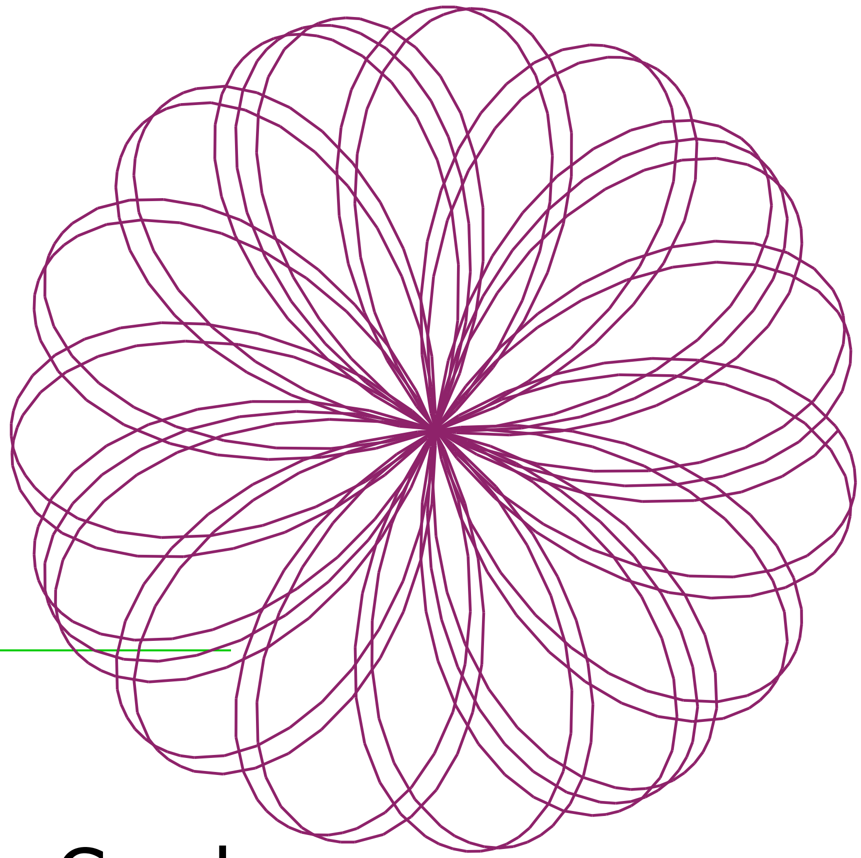
Questions about *Self-Study* Arc Length of Polar Curves

HW Questions

Answers:

55) 16π
56) $2a\pi$
57) 4π
58) $2a\pi$
59) 8
60) 64

Calculus BC

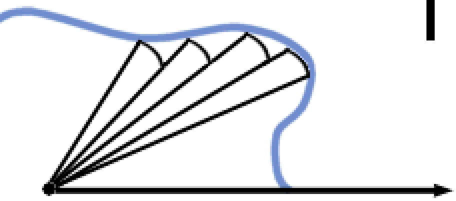


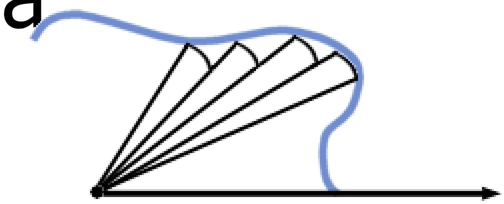
NEW TOPIC

Area **Enclosed** by Polar Graphs

Day 5

Today's topic—Finding Area **Enclosed** by Polar Curve

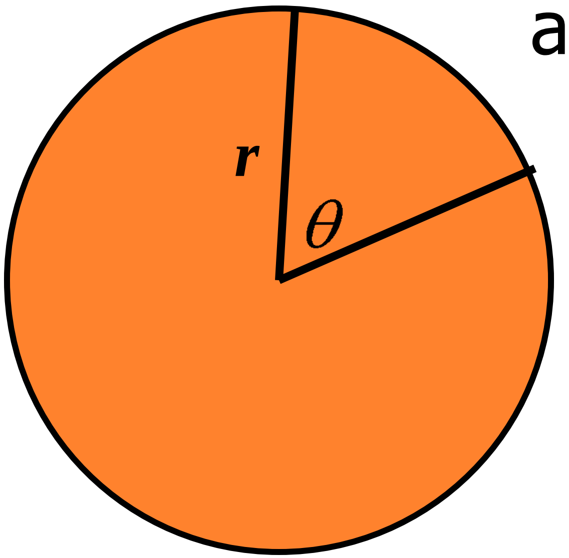




- Note we are finding area "enclosed by" instead of "under" the polar curve!
- Remember in Calculus AB we used ***rectangles*** to approximate the area between a curve and the x-axis or between two curves.
- For polar graphs, we will be using ***sectors*** of a circle to approximate the area enclosed by a polar curve.

From Geometry:

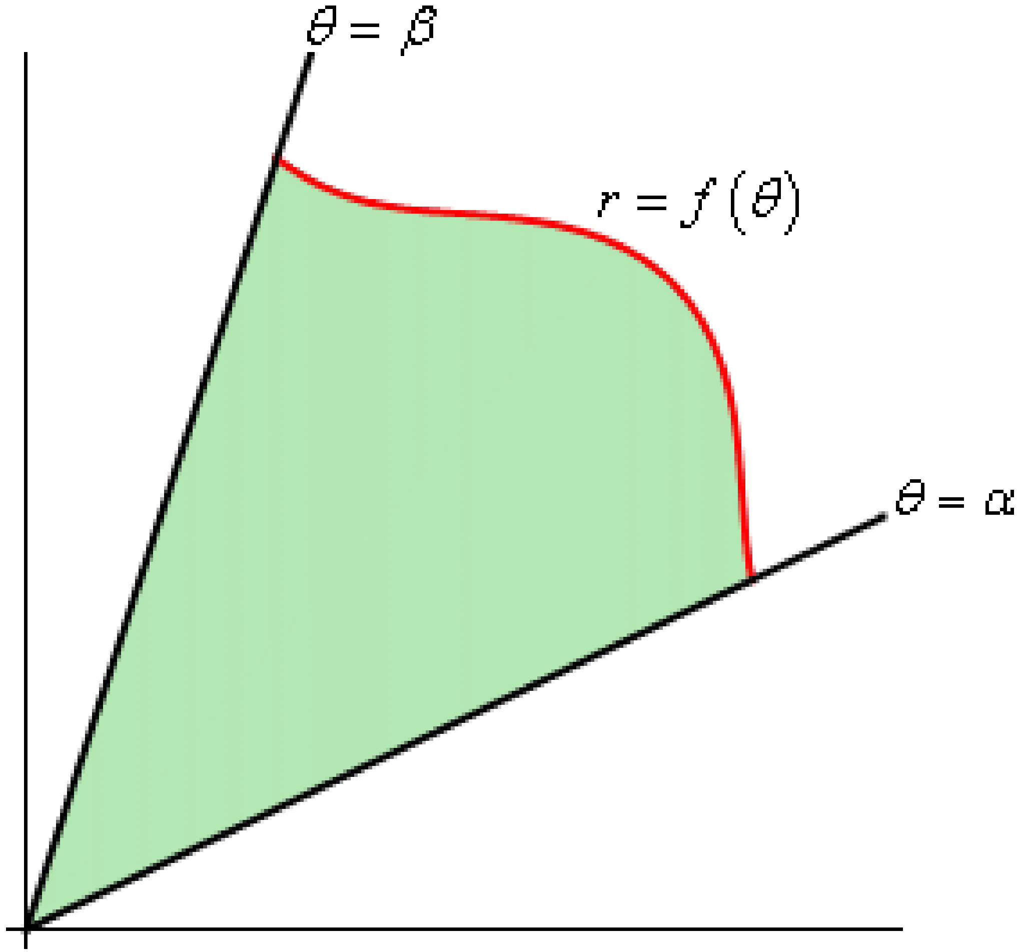
- Given a circle with radius of r.



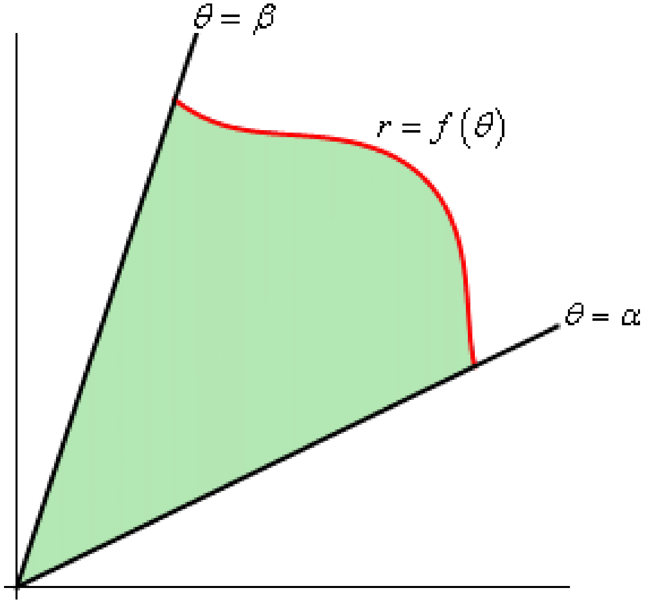
- The area of the sector with central angle θ, measured in radians, is

$$A_{\text{sector}} = \frac{1}{2} r^2 \theta$$

Below is the graph of the polar curve $r = f(\theta)$:



We'll be looking for the shaded area in the sketch above

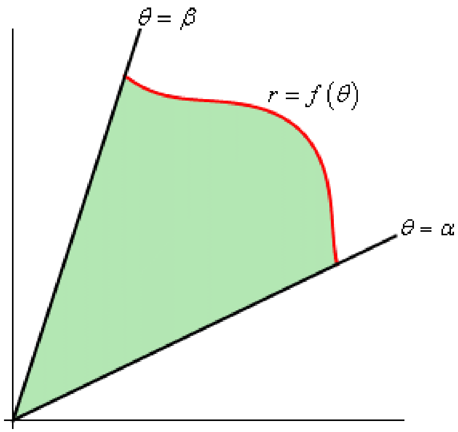


The interval $[\alpha, \beta]$ is divided into n subintervals. The length of each subinterval is $\dfrac{\beta - \alpha}{n}$

Let θ_k be the midpoint of a subinterval.

Construct a circular sector with the center at the origin, radius $r_k = f(\theta_k)$ and central angle $\Delta\theta_k$.

The area of this constructed sector is therefore equal to $A_k = \dfrac{1}{2} r_k^{\,2} \Delta\theta_k$

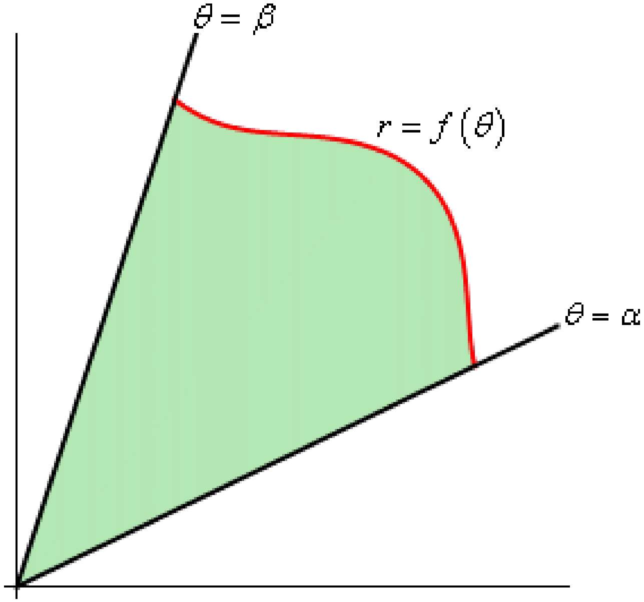


If we repeat this process “n” times then the approximate area of the shaded region would be:

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} r_k{}^2 \Delta\theta_k$$

As the number of subintervals increases, the approximation of the area continues to improve and

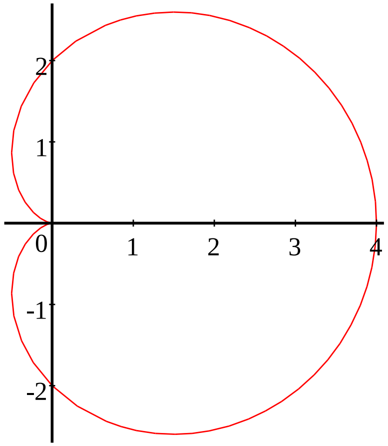
$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2} r_k{}^2 \Delta\theta_k = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$



So, the area of the shaded region can be calculated using

$$A_{enclosed\_by\_polar\_curve} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Example: Find the area enclosed by: $r = 2(1+\cos\theta)$



$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \cdot 4 \left(1+\cos\theta\right)^2 d\theta$$

$$= \int_0^{2\pi} 2\left(1+2\cos\theta+\cos^2\theta\right) d\theta$$

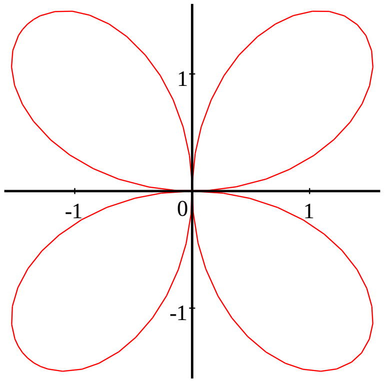
$$= \int_0^{2\pi} 2\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= 2\left(\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\Bigg|_0^{2\pi}\right) = 6\pi$$

Using Symmetry Properties can save you time…

$$r = 2\sin\ 2\theta$$



NOTE

Area of one leaf times 4:

$$A = 4 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[2\sin\ 2\theta\ \right]^2 d\theta$$

Area of four leaves:

$$A = \frac{1}{2}\int_0^{2\pi} \left[2\sin\ 2\theta\ \right]^2 d\theta$$

$$A = 2\pi$$

Pay close attention for multiple choice questions.

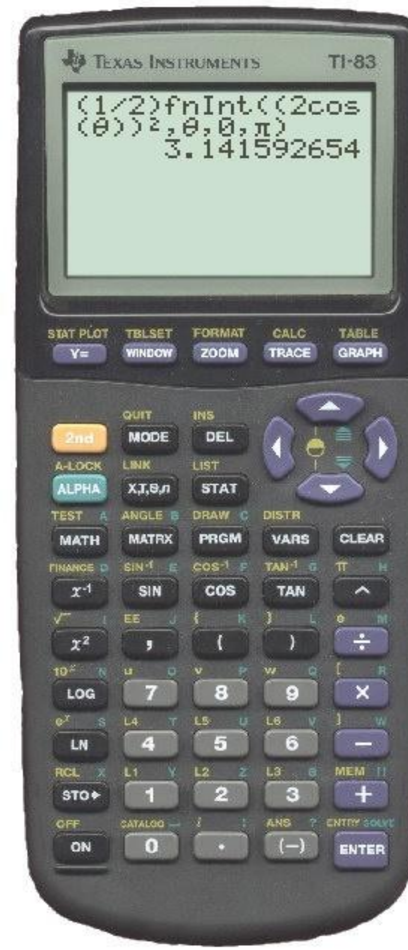
The area bounded by the curve $r = 2\cos\theta$

can be either of the following integrals . . .

NOTE

$$\frac{1}{2}\int_{0}^{\pi} 2\cos\theta\ ^{2}\,d\theta$$

$$or\ (2)\frac{1}{2}\int_{0}^{\pi/2} 2\cos\theta\ ^{2}\,d\theta$$



Check by evaluating on the calculator.

Another Example Problem

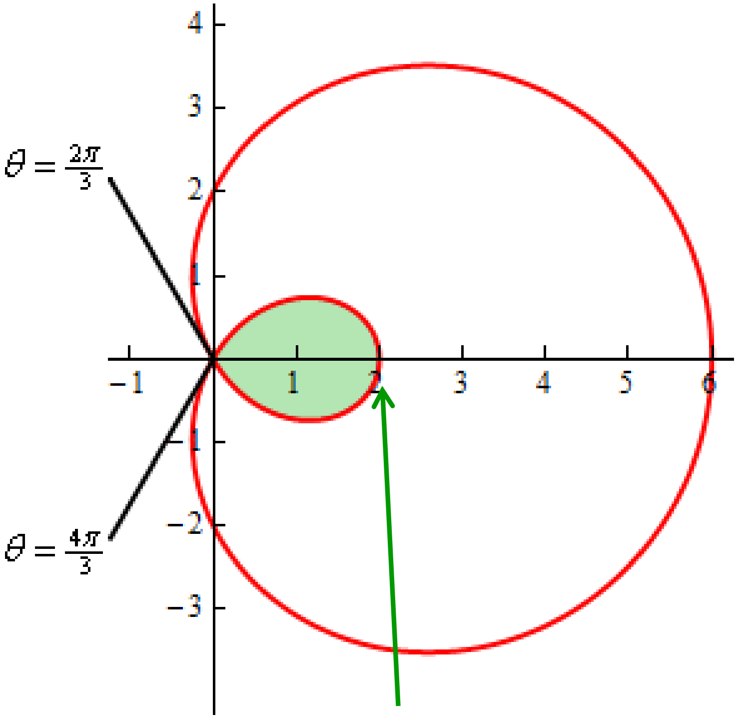
Determine the area of the inner loop of

$$r(\theta) = 2 + 4\cos\theta$$

To do this we will need the θ values that generate the inner loop.

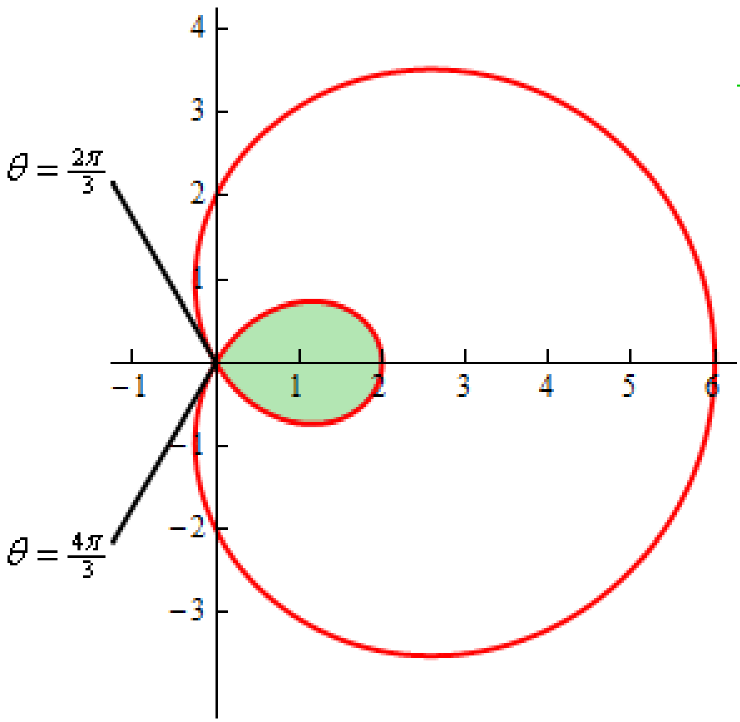
Since we know there is a location on the curve where *r=0,* set the equation equal to zero and solve.

$$2+4\cos\theta=0$$



Checking for understanding . . What are the polar coordinates of this point?

$r(\theta) = 2 + 4\cos\theta$



$$\int_{2\pi/3}^{4\pi/3} \frac{1}{2}\left(2+4\cos\theta\right)^{2} d\theta$$

$$= \int_{2\pi/3}^{4\pi/3} \frac{1}{2}\left(4+16\cos\theta+16\cos^{2}\theta\right) d\theta$$

$$\int_{2\pi/3}^{4\pi/3} \left(2+8\cos\theta+8\cos^{2}\theta\right) d\theta$$

$$= \int_{2\pi/3}^{4\pi/3} \left(2+8\cos\theta+8\left(\frac{1+\cos 2\theta}{2}\right)\right) d\theta = 4\pi - 6\sqrt{3}$$