BC Calculus

Day 5

Taylor & Maclaurin

Series

WARMUP—Calculus AB Review Problems

x	2	5	7	8
$f(x)$	10	30	40	20

C 98-38 The function f is continous on the closed interval [2, 8] and has values that are given in the table above. Using the subintervals [2, 5], [5, 7], and [7, 8], what is the trapezoidal approximation of $\int_2^8 f(x)dx$?

(A) 110 (B) 130 (C) 160 (D) 190 (E) 210

C 98-40 Which of the following is an equation of the line tangent to the graph of $f(x) = x^4 + 2x^2$ at the point where $f'(x) = 1$?

(A) $y = 8x - 5$ (B) $y = x + 7$ (C) $y = x + 0.763$
(D) $y = x - 0.122$ (E) $y = x - 2.146$

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HW Questions

Packet p. 3 (Free Response Practice)

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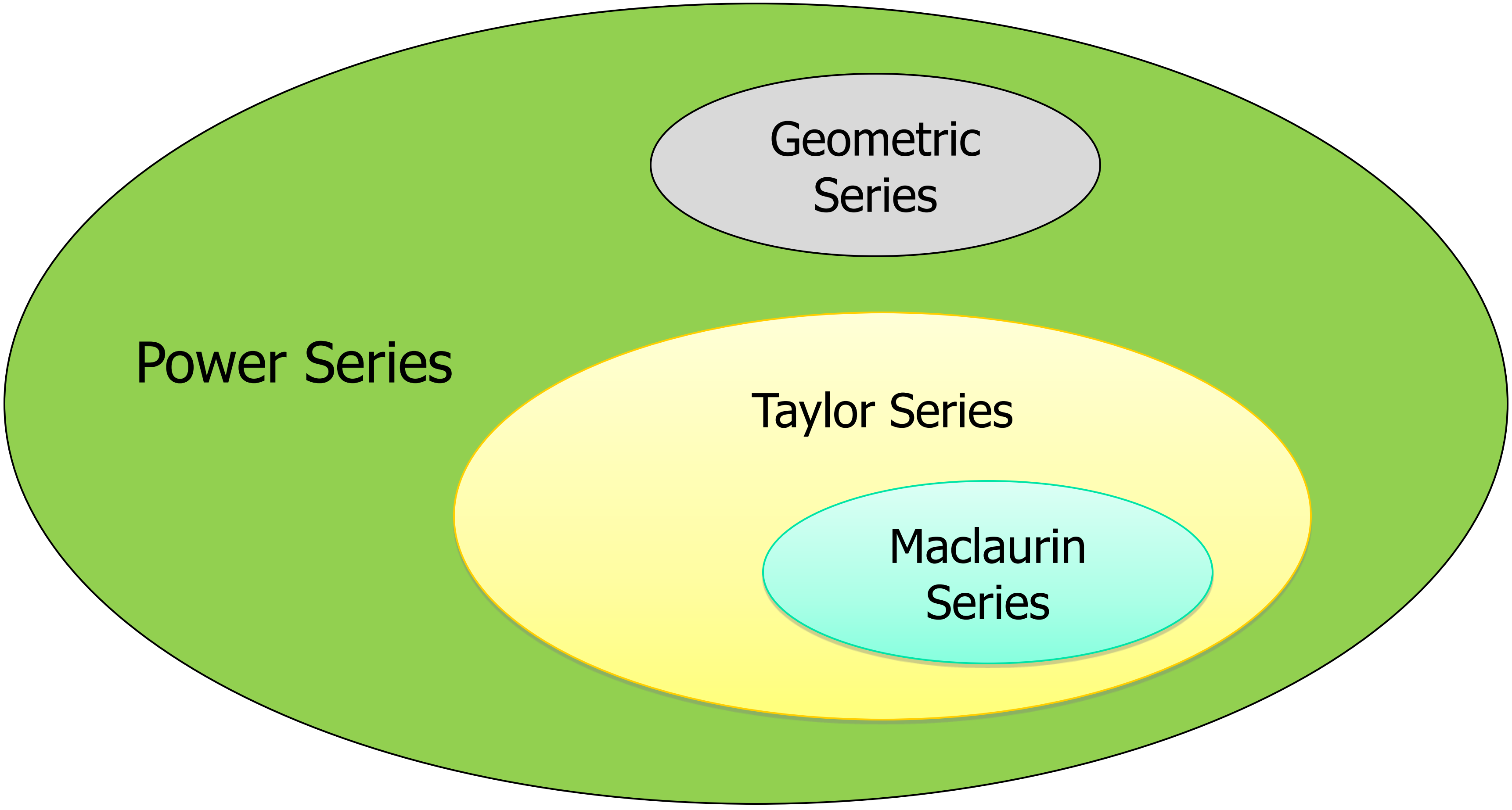
Our story so far



How we have used power series

We found power series representations when functions or their integrals or derivatives were of form $\dfrac{a}{1-r}$

$$f(x)=\frac{1}{1-2x} \qquad f(x)=\tan^{-1}x$$

Today, our goal is the same but our method is different.

Why are we bothering?

$$\cos 0 = 1$$

$$\sqrt[3]{8} = 2$$

$$\ln 1 = 0$$

$$e^1 = e$$

$$\cos 2 = ?$$

$$\sqrt[3]{2} = ?$$

$$\ln 2 = ?$$

$$e^2 = ?$$

Easily memorized and recalled

Series give us the means to approximate the value of functions

Consider $f(x) = \cos x$

This is not in $\dfrac{a}{1-r}$ form, nor is its derivative or integral.

But finding a powers series representation is still worthwhile.

$$\cos x = \sum_{n=0}^{\infty} ?$$

Let's start with our power series template

$$f(x) = \cos x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

Where is the series centered?

Our goal is to solve for all the c_s

Let *x=0*

$$c_0 = \cos 0 \qquad c_0 = 1$$

The result

$$\cos x = 1 + \cdots$$

We have our first term!

Now, let's get creative

$$f(x) = \cos x = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

Getting the x^2 terms and later to disappear:

Take the derivative of each side

$$f'(x) = -\sin x = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + nc_n x^{n-1} + \cdots$$

Let $x = 0$

$$-\sin 0 = c_1 \qquad c_1 = 0$$

$$\cos x = 1 + 0x + \cdots$$

$$f(x) = \cos x = 1 + 0x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

Let's keep going

$$f'(x) = -\sin x = 0 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + nc_n x^{n-1} + \cdots$$

Take the derivative again of each side

$$f''(x) = -\cos x = 2c_2 + 2\cdot 3c_3 x + 3\cdot 4c_4 x^2 + \cdots + (n-1)nc_n x^{n-2}$$

Let $x = 0$.

$$-\cos 0 = 2c_2 \qquad c_2 = -\frac{1}{2}$$

$$\cos x = 1 + 0x - \frac{1}{2}x^2 + \cdots$$

$$f(x) = \cos x = 1 + 0x - \frac{x^2}{2} + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

Again

$$f''(x) = -\cos x = 2\left(-\frac{1}{2}\right) + 2\cdot 3c_3 x + 3\cdot 4c_4 x^2 + \cdots + (n-1)nc_n x^{n-2}$$

Take the derivative again of each side

$$f'''(x) = \sin x = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4 x + \cdots + (n-2)(n-1)nc_n x^{n-3}$$

Let $x = 0$.

$$\sin 0 = 2\cdot 3\cdot c_3 \qquad c_3 = 0$$

$$\cos x = 1 + 0x - \frac{x^2}{2} + 0x^3 + \cdots$$

$$f(x) = \cos x = 1 + 0x - \frac{x^2}{2} + 0x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

Yep, again

$$f'''(x) = \sin x = 0 + 2 \cdot 3 \cdot 4 c_4 x + \cdots + (n-2)(n-1) n c_n x^{n-3}$$

$$f^4(x) = \cos x = 2 \cdot 3 \cdot 4 c_4 + \cdots + (n-3)(n-2)(n-1) n c_n x^{n-4}$$

Let $x = 0$.

$$\cos 0 = 2 \cdot 3 \cdot 4 c_4 \qquad c_4 = \frac{1}{2 \cdot 3 \cdot 4} = \frac{1}{4!}$$

$$\cos x = 1 + 0x - \frac{x^2}{2} + 0x^3 + \frac{x^4}{4!} + \cdots$$

Continuing we would get . . . . .

$$\cos x = 1 + 0x - \frac{x^2}{2} + 0x^3 + \frac{x^4}{4!} + 0x^5 - \frac{x^6}{6!} + 0x^7 + \frac{x^8}{8!} \cdots$$

What is the general pattern in terms of *n*?

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots$$

n=0 n=1 n=2 n=3 n=4

nth term . . . .

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots + \left(-1\right)^{?}\frac{x^{?}}{?} + \cdots$$

n=0 n=1 n=2 n=3 n=4

NOTE the +. . .

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \ -1 \ ^{n} \frac{x^{2n}}{2n \ \ !} + \cdots = \sum_{n=0}^{\infty} \ -1 \ ^{n} \frac{x^{2n}}{2n \ \ !}$$

PUT THIS in YOUR NOTES

Now we can appoximate the value of *f(2)=cos(2)*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$f(2) = \cos 2 = 1 - \frac{2^2}{2} + \frac{2^4}{4!} - \frac{2^6}{6!} + \cdots + (-1)^n \frac{2^{2n}}{(2n)!} + \cdots$$

NOTE: The more terms of the series that are used, the better the approximation.

What is the interval of convergence?

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

0

$$\lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\left|x^2\frac{1}{(2n+1)(2n+2)}\right| = 0$$

$0 < 1$ Regardless of what x equals

$\therefore$ Series converges for all real #s

Awesome?

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

For all real #s

And all we needed to know was

how $\cos x$ behaves at $x = 0$.

Generalizing for a power series centered at any *x=a* not just 0

Power series centered at $x = a$.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

$$f'(x) = c_1 + 2c_2(x-a) + \cdots + (n)c_n(x-a)^{n-1} + \cdots$$

$$f''(x) = 2c_2 + \cdots + (n-1)(n)c_n(x-a)^{n-2} + \cdots$$

$$f'''(x) = \cdots + (n-2)(n-1)(n)c_n(x-a)^{n-3} + \cdots$$

$$f^n(x) = (1)(2)\cdots(n-3)(n-2)(n-1)(n)c_n + \cdots$$

$$f^n(x) = n!c_n + \cdots$$

$$c_n = \frac{f^n(a)}{n!}$$

We have a formula to determine each coefficient.

Think about what is happening

Every time we take a derivative,

The first degree term becomes just a constant.

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

The leading constant goes away

The rest of the terms go away when we let $x = 0$.

Definition: Taylor Series

if *f* is a function with derivatives of all orders throughout some open interval containing *a*, then:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$

$$= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^{2}+\frac{f'''(a)}{3!}(x-a)^{3}+\ldots$$

0! = 1, so the first term will always end up being f(a).

A Taylor Series centered at ***a* = 0** is known as a **Maclaurin Series**.

Ex) Find the Maclaurin series for *f(x) = e^x^*

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$

n	$f^{n}(x)$	$f^{n}(a)=f^{n}(\_\_\_)$
0	$f(x)=e^{x}$	$f(0)=1$
1	$f'(x)=e^{x}$	$f'(0)=1$
2	$f''(x)=e^{x}$	$f''(0)=1$
3	$f'''(x)=e^{x}$	$f'''(0)=1$
4	$f^{4}(x)=e^{x}$	$f^{4}(0)=1$

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$$

Let's build our series

NOTE the +. . .

$$e^x=1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Now approximate the value of *f(5)*

NOTE the +. . .

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(5) = e^5: \qquad e^5 = 1 + 5 + \frac{5^2}{2} + \frac{5^3}{3!} + \cdots$$

NOTE: The more terms of the series that are used the better the approximation.

To find the interval of convergence, do Ratio Test on $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

0

$$= \lim_{n\to\infty}\left|x\frac{1}{n+1}\right| = 0 < 1$$

The series converges for all real #s and the radius of convergence is R = ∞

And we only needed to know behavior at *x* = 0

Another Ex) Find a power series expansion for *f(x)*=ln(x) centered at 1.

n	$f^{n}(x)$	$f^{n}(a) = f^{n}(\_\_\_)$
0	$f(x) = \ln x$	$f(1) = 0$
1	$f'(x) = \frac{1}{x}$	$f'(1) = 1$
2	$f''(x) = -\frac{1}{x^2}$	$f''(1) = -1$
3	$f'''(x) = \frac{2}{x^3}$	$f'''(1) = 2!$
4	$f^{4}(x) = -\frac{2\cdot 3}{x^4} = \frac{3!}{x^4}$	$f^{4}(1) = -2\cdot 3 = -3!$
5	$f^{5}\ \ x\ \ = \frac{2\cdot 3\cdot 4}{x^5} = \frac{4!}{x^5}$	$f^{5}\ \ 1\ \ = 2\cdot 3\cdot 4 = 4!$

$$f(1)=0$$
$$f'(1)=1$$
$$f''(1)=-1$$
$$f'''(1)=2!$$
$$f^{4}(1)=-3!$$
$$f^{5}\ 1\ =4!$$

Now let's write our series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$

$$f(x)=0+1(x-1)-\frac{(x-1)^{2}}{2!}+\frac{2!(x-1)^{3}}{3!}-\frac{3!(x-1)^{4}}{4!}+\ldots$$

$$f(x)=(x-1)-\frac{(x-1)^{2}}{2}+\frac{(x-1)^{3}}{3}-\frac{(x-1)^{4}}{4}+\cdots+(-1)^{n-1}\frac{(x-1)^{n}}{n}+\cdots$$

Find the Interval of Convergence

$$\ln\ x-1\ =\sum\ -1^{\ n-1}\frac{x-1^{\ n}}{n}$$

$$\lim_{n\to\infty}\left|\frac{x-1^{\ n+1}}{n+1}\cdot\frac{n}{x-1^{\ n}}\right|=\lim_{n\to\infty}\left|\ x-1\ \cdot\frac{n}{n+1}\right|=\left|\ x-1\ \right|$$

$$-1<x-1<1$$

$$0<x<2$$

Memorize These

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{all real \#s}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{all real \#s}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad \text{all real \#s}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \qquad -1 < x \le 1$$

Use a Maclaurin series derived in this section to find a Maclaurin series for the following . . . .

Find the Maclaurin series for $\frac{(1+\cos 2x)}{2}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad \text{(from previous slide)}$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots$$

$$1 + \cos 2x = 2 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots$$

$$\frac{1+\cos 2x}{2} = \frac{2}{2} - \frac{(2x)^2}{2!2} + \frac{(2x)^4}{4!2} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!2} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad g(x) = \frac{e^x - 1}{x^2}$$

Find the 1st three terms of a series for g(x) and the nth term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 = \quad x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{e^x - 1}{x^2} = \frac{x}{x^2} + \frac{x^2}{x^2 2!} + \frac{x^3}{x^2 3!} + \cdots = x^{-1} + \frac{1}{2!} + \frac{x}{3!} + \cdots + \frac{x^{n-1}}{(n+1)!} + \cdots$$

PRACTICE . . . .