Unit 3 Day 4
Wrap up of Improper Integrals

Warmup—Calculus AB Practice

1. If $f(x) = \sin(e^{-x})$ then find $f'(x)$.

2. Write the equation of the tangent line to the curve $y = x + \cos x$ at the point $(0,1)$

HW Questions?

Before we go over hw let's recap what we did yesterday . . . . . .

Fundamental Theorem of Calculus only works for integrands that are <u>continuous</u> functions over <u>closed</u> intervals.

$\int_a^b f(x)\,dx = F(b) - F(a)$

Definite Integral

$\int f(x)\,dx = F(x) + c$

Indefinite Integral

Yesterday we learned about improper integrals.

Infinite Discontinuity at lower bound

$$\int_0^5 \frac{1}{x}\,dx = \lim_{a\to 0^+}\int_a^5 \frac{1}{x}\,dx$$

Infinite Discontinuity at upper bound

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx = \lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt{1-x}}dx$$

Infinite Discontinuity in between upper/lower bound

$$\int_{-1}^{2} \frac{1}{x^2}dx = \int_{-1}^{0} \frac{1}{x^2}dx + \int_{0}^{2} \frac{1}{x^2}dx = \lim_{b\to 0^-}\int_{-1}^{b} \frac{1}{x^2}dx + \lim_{a\to 0^+}\int_{a}^{2} \frac{1}{x^2}dx$$

These integrals can either:
1. DIVERGE—grow and grow
or
2. CONVERGE to a particular number

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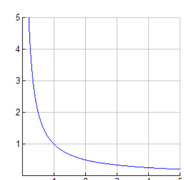
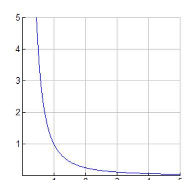
HW Questions??

Improper Integrals
One more category . . .
Infinite Limit of Integration

$$\int_1^\infty \frac{1}{x}\,dx \qquad \int_1^\infty \frac{1}{x^2}\,dx$$



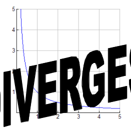


One of these converges, the other diverges

Example 1a:

$$\int_1^{\infty} \frac{1}{x}\,dx$$

The function accumulates area without bound.



This integral DIVERGES

We take the integral as usual, but we can't just substitute infinity.

We must approach infinity.

Can we find the area of such a curve?

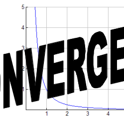
$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1) = \infty$$

Example 1b:

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$

The function accumulates area without bound.



This integral CONVERGES

We take the integral as usual, but we can't just substitute infinity.

We must approach infinity.

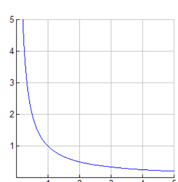
Can we find the area of such a curve?

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\,dx$$

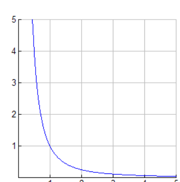
$$= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} - (-\frac{1}{1})\right) = 1$$

You cannot just look at the graph and tell

$$\int_1^{\infty} \frac{1}{x}\,dx \quad Diverges$$



$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1$$



CW--Practice

Evaluate the following or state that it diverges

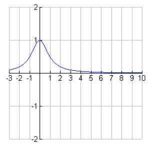
1. $\int_0^1 \frac{dx}{\sqrt[3]{x^2}}$

2. $\int_0^{\infty} \frac{1}{1+x^2}dx$ hint: Think inverse trig!

Solution on next slide. ☺

Example 2:

$\int_0^{\infty} \frac{1}{1+x^2} dx$ ← The function also accumulates area without bound.



$$\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2} dx$$

This interval CONVERGES to

$$= \lim_{b\to\infty} \tan^{-1} x \Big|_0^b = \lim_{b\to\infty} \left(\tan^{-1}(b) - \tan^{-1}(0)\right)$$

$$= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

Here is a more challenging problem to try:

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx$$

HINT: Rationalize the numerator. Then split the result to have two expressions.

$$\int_0^1 \sqrt{\frac{1+x}{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\, dx = \int_0^1 \frac{1+x}{\sqrt{1-x^2}}\, dx$$

$$\lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\, dx + \lim_{b\to 1^-} \int_0^b \frac{x}{\sqrt{1-x^2}}\, dx$$

→

$$\int \frac{1}{\sqrt{1-x^2}}\, dx + \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$$u = 1-x^2$$
$$du = -2x\, dx$$
$$-\frac{1}{2} du = x\, dx$$

$$\sin^{-1} x - \frac{1}{2} \int u^{-\frac{1}{2}} du$$

$$\sin^{-1} x - u^{\frac{1}{2}}$$

$$\lim_{b\to 1^-} \sin^{-1} x - \sqrt{1-x^2} \Big|_0^b$$

This integral converges because it approaches a solution.

$$\lim_{b\to 1^-} \left(\underbrace{\sin^{-1} b}_{\frac{\pi}{2}} - \underbrace{\sqrt{1-b^2}}_{0}\right) - \left(\underbrace{\sin^{-1} 0}_{0} - \sqrt{1}\right) = \boxed{\frac{\pi}{2} + 1}$$

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