Unit 2 Parametrics

Day 3 - Vector Valued Functions

Review of position, velocity, and acceleration

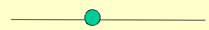
s(t) = Position at time *t*

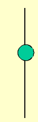
v(t) = Velocity at time *t*

a(t) = Acceleration at time *t*

$$v(t) = \frac{ds}{dt}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$





Models motion in 1-dimension

Position, velocity, and acceleration in 2-dimensional space

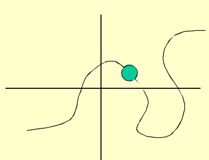
r(t) = Position at time *t*

v(t) = Velocity at time *t*

a(t) = Acceleration at time *t*

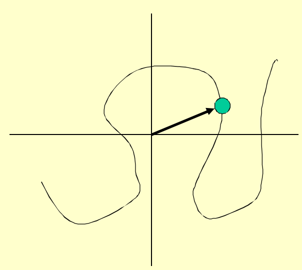
$$v(t) = \frac{dr}{dt}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2 r}{dt^2}$$



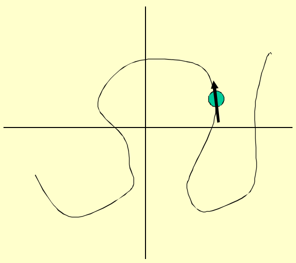
Same idea, but we need way to deal with 2-dimensions

Enter Vectors



Position vector – position of object at time *t*, relative to the origin.

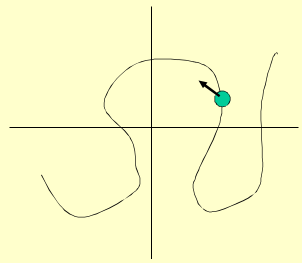
Enter Vectors



Position vector – position of object at time *t,* relative to the origin.

Velocity vector – direction and speed of object at time *t.*

Enter Vectors

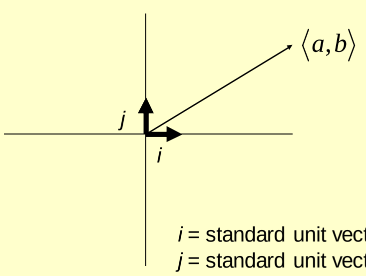


Position vector – position of object at time *t,* relative to the origin.

Velocity vector – direction and speed of object at time *t.*

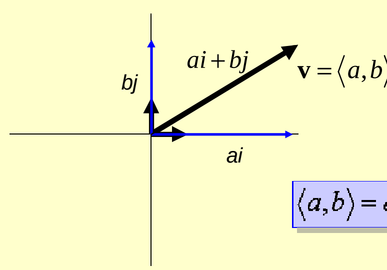
Acceleration vector – how speed and direction are changing at time *t.*

Standard Unit Vectors



i = standard unit vector <1,0>
j = standard unit vector <0,1>

Any vector $\mathbf{v}=\langle a,b\rangle$ can be written as a linear combination of the two standard unit vectors.



$$\langle a,b\rangle = ai + bj$$

Here is the proof:

$\mathbf{i} = \langle 1,0 \rangle \qquad \mathbf{j} = \langle 0,1 \rangle$

Start with

$$\begin{aligned} \mathbf{v} &= \langle a,b \rangle \\ &= \langle a,0 \rangle + \langle 0,b \rangle \\ &= a\langle 1,0 \rangle + b\langle 0,1 \rangle \\ &= a\mathbf{i} + b\mathbf{j} \end{aligned}$$

The vector **v** is now a linear combination of the vectors **i** and **j**.

The scalar a is the horizontal component of **v** and the scalar b is the vertical component of **v**.

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Ex. Let P=(-1, 5) and Q = (3,2). Write $\overline{PQ}$ as a linear combination of **i + j**

$$\overrightarrow{PQ} = \langle 4,-3 \rangle = 4i - 3j$$

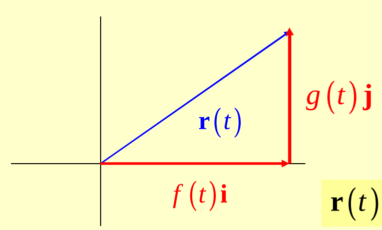
Understanding 2D Motion

A particle moving through a plane:

The particle's coordinates are a function of time. Ex: $x=f(t)$ and $y=g(t)$

The set of points (x,y) make the curve in the plane that represents the particle's path.

We can describe the position of a moving particle by a vector, $r(t)$.



$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$

If we separate $r(t)$ into horizontal and vertical components, we can express $r(t)$ as a linear combination of standard unit vectors **i** and **j**.

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SPEED vs VELOCITY

$$\text{Speed} = |\mathbf{v}(\mathbf{t})|$$

"Speed" is magnitude of velocity.
Speed has no direction.
Velocity has direction.

$$\text{Direction of motion} = \frac{\text{velocity vector}}{\text{speed}} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

"Direction" is a unit vector that indicates direction but not magnitude.

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Example: Given *r(t)* is the position vector of a particle at time *t:*

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$$

a) Find the velocity and acceleration vectors.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$$

Example: $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$$

b) Find the velocity and acceleration $t = \pi/4$

velocity: $\mathbf{v}\left(\frac{\pi}{4}\right) = \left(-3\sin\frac{\pi}{4}\right)\mathbf{i} + \left(3\cos\frac{\pi}{4}\right)\mathbf{j} = -\frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$

acceleration: $\mathbf{a}\left(\frac{\pi}{4}\right) = \left(-3\cos\frac{\pi}{4}\right)\mathbf{i} - \left(3\sin\frac{\pi}{4}\right)\mathbf{j} = -\frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$

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Example: $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$$

c) Find the speed and direction of motion at $t = \pi/4$.

From Part b:

$$\mathbf{v}\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j} \qquad \mathbf{a}\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$$

speed: $\left|\mathbf{v}\left(\frac{\pi}{4}\right)\right| = \sqrt{\left(-\frac{3\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{2}}{2}\right)^2} = \sqrt{\frac{18}{4} + \frac{18}{4}} = 3$

direction: $\frac{\mathbf{v}(\pi/4)}{|\mathbf{v}(\pi/4)|} = \frac{-3\sqrt{2}/2}{3}\mathbf{i} + \frac{3\sqrt{2}/2}{3}\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

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Let's explore:

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}\Big|_{t=\frac{\pi}{4}} \Rightarrow \quad r\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j}\Big|_{t=\frac{\pi}{4}} \Rightarrow \quad \mathbf{v}\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j}\Big|_{t=\frac{\pi}{4}} \Rightarrow \quad a\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$$

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Let's explore:

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}\Big|_{t=\frac{\pi}{4}} \Rightarrow \quad r\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j}\Big|_{t=\frac{\pi}{4}} \Rightarrow \quad \mathbf{v}\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j}\Big|_{t=\frac{\pi}{4}} \Rightarrow \quad a\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$$

Notice . . . .

Velocity is $\perp$ to position.

Acceleration is $\perp$ to velocity and opposite of position.

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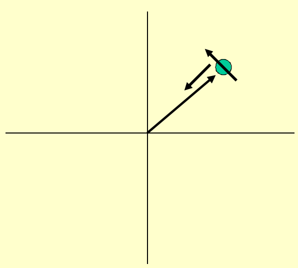
$$r\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{v}\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

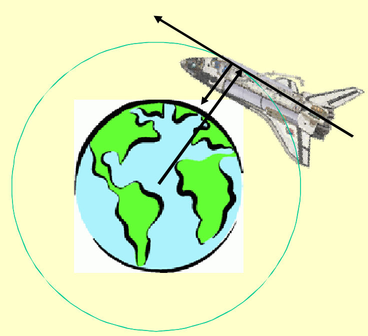
$$a\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$$



Velocity is $\perp$ to position.

Acceleration is $\perp$ to velocity and opposite of position.

This is a unique property of sine and cosine.



Example: Given *r(t)* is the position vector of a particle at time *t:*

$$\mathbf{r}(t)=\left(2t^3-3t^2\right)\mathbf{i}+\left(t^3-12t\right)\mathbf{j}$$

a) Write the equation of the tangent where $t=-1$.

At $t=-1$: $\mathbf{r}(-1)=-5\mathbf{i}+11\mathbf{j}$

point: $(-5,11)$

$slope=\frac{d\mathbf{r}}{dt}=\left(6t^2-6t\right)\mathbf{i}+\left(3t^2-12\right)\mathbf{j}$

$slope=12\mathbf{i}-9\mathbf{j}$

slope: $\frac{-9}{12}=-\frac{3}{4}$

tangent: $y-y_1=m\left(x-x_1\right)$

$y-11=-\frac{3}{4}(x+5)$

$y=-\frac{3}{4}x+\frac{29}{4}$

Example:

$$\mathbf{r}(t)=\left(2t^3-3t^2\right)\mathbf{i}+\left(t^3-12t\right)\mathbf{j}$$

$$\mathbf{v}(t)=\frac{d\mathbf{r}}{dt}=\left(6t^2-6t\right)\mathbf{i}+\left(3t^2-12\right)\mathbf{j}$$

b) Find the coordinates of each point on the path where the **horizontal COMPONENT** of the velocity is 0.

The horizontal component of the velocity is $6t^2-6t$.

$6t^2-6t=0$

$t^2-t=0$

$t(t-1)=0$

$t=0,1$

$\mathbf{r}(0)=0\mathbf{i}+0\mathbf{j}$ ➡ $(0,0)$

$\mathbf{r}(1)=(2-3)\mathbf{i}+(1-12)\mathbf{j}$

$\mathbf{r}(1)=-1\mathbf{i}-11\mathbf{j}$ ➡ $(-1,-11)$

π

AP Calculus BC **2003Free Response**

BC 4 (B)

4. A particle moves in the *xy*-plane so that the position of the particle at any time *t* is given by
$x(t)=2e^{3t}+e^{-7t}$ and $y(t)=3e^{3t}-e^{-2t}$

(a) Find the velocity vector for the particle in terms of *t*, and the speed of the particle at time $t=0$.

(b) Find $\frac{dy}{dx}$ in terms of *t*, and find $\lim_{t\to\infty}\frac{dy}{dx}$.

(c) Find each value *t* at which the line tangent to the path of the particle is horizontal, or explain why none exists.

(d) Find each value *t* at which the line tangent to the path of the particle is vertical, or explain why none exists.

(a) $x'(t)=6e^{3t}-7e^{-7t}$ $y'(t)=9e^{3t}+2e^{-2t}$ Velocity vector is $<6e^{3t}-7e^{-7t},\ 9e^{3t}+2e^{-2t}>$ Speed $=\sqrt{x'(0)^2+y'(0)^2}=\sqrt{(-1)^2+11^2}$ $=\sqrt{122}$	3 : { 1 : $x'(t)$; 1 : $y'(t)$; 1 : speed
(b) $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{9e^{3t}+2e^{-2t}}{6e^{3t}-7e^{-7t}}$ $\lim_{t\to\infty}\frac{dy}{dx}=\lim_{t\to\infty}\frac{9e^{3t}+2e^{-2t}}{6e^{3t}-7e^{-7t}}=\frac{9}{6}=\frac{3}{2}$	2 : { 1 : $\frac{dy}{dx}$ in terms of t; 1 : limit

c) Need $y'(t) = 0$, but $9e^{3t} + 2e^{-2t} > 0$ for all t, so none exists.	$2: \begin{cases} 1: \text{considers } y'(t) = 0 \\ 1: \text{explains why none exists} \end{cases}$
d) Need $x'(t) = 0$ and $y'(t) \neq 0$. $6e^{3t} = 7e^{-7t}$ $e^{10t} = \frac{7}{6}$ $t = \frac{1}{10}\ln\left(\frac{7}{6}\right)$	$2: \begin{cases} 1: \text{considers } x'(t) = 0 \\ 1: \text{solution} \end{cases}$