Unit 3 Improper Integrals

Day 3

Warmup—A step into the unknown?

Evaluate the following limit:

$$\lim_{x \to 1} \frac{\int_1^x \sin t \, dt}{x^2 - 1}$$

Brainstorm with your partner. ☺

Use your calculator to find the final answer.

HW Questions

Volunteer to help others or put a problem on the board.

Stuck? Ask for help!

Today's Topic: Improper Integrals

Unit 3 Day 3

Remember the Fundamental Theorem of Calculus . . . .

$$\int_a^b f(x)dx = F(b) - F(a)$$

But do you *really* remember the Fundamental Theorem of Calculus . . . .

- There is something "fundamentally" wrong with the following:

$$\int_{-1}^{2} \frac{1}{x^2}dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{2} = -\frac{3}{2}$$

- Discuss with your partner why this seems suspicious. Consider the graph of $f(x) = \frac{1}{x^2}$

during your discussion.

Suspicious answer . . .

- An answer of -3/2 should seem incorrect since the integrand $1/x^2$ is always positive
- The issue is that the hypothesis of the Fundamental Theorem of Calculus requires the integrand to be continuous at every point on the interval of integration.
- $f(x) = \frac{1}{x^2}$ is discontinuous at x=0 which is within the interval [-1,2]

This previous example is one type of "Improper Integral"

- Recall, the definition of a definite integral is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

where c_i is a point in a subinterval of [a,b]

- If $f(x) \to \infty$ at some point in [a,b] then the limit defining $\int_a^b f(x)dx$ is meaningless.

NOTES

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx$$

Improper because the integrand is discontinuous at x=1, which is the upper bound for the integration

The official definition

If *f* is continuous on the interval [a,b) and $f(x) \to \pm\infty$ as $x \to b^-$ then the improper integral of *f* on [a,b] is defined to be:

$$\int_a^b f(x)dx = \lim_{R \to b^-} \int_a^R f(x)dx$$

Likewise . . .

If *f* is continuous on the interval (a,b] and $f(x) \to \pm\infty$ as $x \to a^+$ then the improper integral of *f* on [a,b] is defined to be:

$$\int_a^b f(x)dx = \lim_{R \to a^+} \int_R^b f(x)dx$$

For both cases . . . .

- If the limit exists (and equals some value L) the improper integral **converges** to L
- If the limit goes to infinity the improper integral **diverges**—continues to grow and grow

Next Example:
(infinite discontinuity at lower bound of integration)

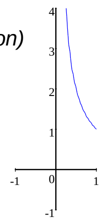
$$\int_0^1 \frac{dx}{x}$$

$$\lim_{a\to0^+}\int_a^1 \frac{1}{x}\,dx \;=\lim_{a\to0^+} \ln x\Big|_a^1$$

$$\lim_{a\to0^+} \ln 1-\ln a$$

$$=0-(-\infty)$$

$$\boxed{=\infty}$$



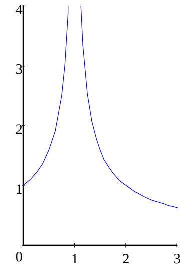
We must approach the discontinuity from <u>inside</u> the interval.

The integral $\int_0^1 \frac{dx}{x}$ <u>diverges</u>.

Another type. *(discontinuity between lower and upper bound)*

$$\int_0^3 (x-1)^{-\frac{2}{3}}\,dx$$

$$=\int_0^1 (x-1)^{-\frac{2}{3}}\,dx+\int_1^3 (x-1)^{-\frac{2}{3}}\,dx$$



$$\lim_{c\to1^-}\int_0^c (x-1)^{-\frac{2}{3}}\,dx+\lim_{c\to1^+}\int_c^3 (x-1)^{-\frac{2}{3}}\,dx$$

$$\lim_{c\to1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^c+\lim_{c\to1^+} 3(x-1)^{\frac{1}{3}}\Big|_c^3$$

$$\lim_{c\to1^-}\int_0^c (x-1)^{-\frac{2}{3}}\,dx+\lim_{c\to1^+}\int_c^3 (x-1)^{-\frac{2}{3}}\,dx$$

$$\lim_{c\to1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^c+\lim_{c\to1^+} 3(x-1)^{\frac{1}{3}}\Big|_c^3$$

$$\lim_{c\to1^-}\left[3\cancelto{0}{(c-1)^{\frac{1}{3}}}-3(-1)^{\frac{1}{3}}\right]+\lim_{c\to1^+}\left[3\cdot 2^{\frac{1}{3}}-3\cancelto{0}{(c-1)^{\frac{1}{3}}}\right]$$

This integral CONVERGES to...

$$\boxed{3+3\sqrt[3]{2}}$$

→

Coming Soon:
One more category . . .

- Improper Integrals with an Infinite Limit of Integration