BC Calculus

Day 3

Derivatives and Integrals of Power Series

WARMUP

Find a power series to represent the given function and identify the interval of convergence

1. $f(x)=\dfrac{1}{1+3x}$

2. $f(x)=\dfrac{x}{1-2x}$

3. $f(x)=\dfrac{3}{1-x^3}$

Rules for simplification

- Pull out (-1) if alternating series
- Combine common bases into single exponential
- Separate coefficient part from variable
- Adjust exponents and series bound to get x^n
- Adjust (-1) exponent to get *n-1, n, or n+1*

$$\sum_{n=0}^{\infty}\frac{x^3}{2}\left(-\frac{x}{2}\right)^n \longrightarrow =\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3} \longrightarrow =\sum_{n=3}^{\infty}\frac{(-1)^{n-3}}{2^{n-2}}x^n$$

$$=\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{2^{n-2}}x^n$$

HW Questions

A Clarification

Are the two series equivalent?

$$\sum_{n=0}^{\infty}x^{n+1} \qquad \sum_{n=1}^{\infty}x^n$$

$$=x+x^2+x^3+\cdots \qquad =x+x^2+x^3+\cdots$$

They generate the same sequence, they are equivalent

A Clarification

Are the two series equivalent?

$$\sum_{n=0}^{\infty}x^{2n+1} \qquad \sum_{n=1}^{\infty}x^{2n}$$

$$=x+x^3+x^5+\cdots \qquad =x^2+x^4+x^6\cdots$$

They DO NOT generate the same sequence, they are NOT equivalent

Moral

When adjusting and simplifying series, you must always ensure the same sequence is generated.

$$\sum_{n=0}^{\infty} x^{n+1} \qquad \sum_{n=1}^{\infty} x^{n}$$

$$= x + x^2 + x^3 + \cdots \qquad = x + x^2 + x^3 + \cdots$$

How do you know if your summation is correct?

$$1 - x^2 + x^4 - \cdots \overset{?}{=} \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Expand out the first few terms of your series and verify you get the same list of terms.

$n=0$	$n=1$	$n=2$
$(-1)^0 x^{2(0)}$	$(-1)^1 x^{2(1)}$	$(-1)^2 x^{2(2)}$
$=1\cdot 1$	$=-1\cdot x^2$	$=1\cdot x^4$
$=1$	$=-x^2$	$=x^4$

Write the first four terms & the nth term of the following function

REMEMBER: Easy and Hard are Relative

First Step: Write function in Sigma Notation $\sum_{n=0}^{\infty} a(r)^n$

Second Step: Use Geometric Series Written in Expanded Form

$$a + ar + ar^2 + \cdots ar^n + \cdots$$

1. $f(x) = \dfrac{1}{1+x^2} =$

$a =$ _____

$r =$ _____

$$\sum_{n=0}^{\infty} a(r)^n = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + \cdots (-1)^n x^{2n} + \cdots$$

Write the first four terms and the nth term of the following functions . . . .

2. $f(x) = \dfrac{1}{2+x}$

$a = \dfrac{1}{2}$

$r = -\dfrac{x}{2}$

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} \cdots (-1)^n \frac{x^n}{2^{n+1}} + \cdots$$

How many terms do you write?

The problem will usually tell you how many.

Example:

"Write the first 4 terms and the nth term"

$$1 - x^2 + x^4 - x^6 + \cdots (-1)^n x^{2n} + \cdots$$

Make sure you remember the "+…"

If the problem does not specify, write 3 or 4 terms.

Helpful Tip. . .
Unless told otherwise, use expanded form instead of summation notation

BENEFIT:

No need to worry about the starting value of "n"

THE DIFFICULTY ("relatively speaking" ☺):

Writing the n^{th} term might be challenging when just looking at a pattern of numbers.

$$\frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} - \frac{x^6}{16} + \cdots (-1)^n \frac{x^{n+3}}{2^{n+1}} + \cdots$$

Practice (Whiteboards or Notes):

Write the first 4 terms and the nth term for the power series to represent the given function and identify the interval of convergence.

$$1.\ f(x)=\frac{1}{1+3x}=1-3x+9x^2-27x^3+\cdots(-1)^n 3^n x^n+\cdots,\quad -\frac{1}{3}<x<\frac{1}{3}$$

$$2.\ f(x)=\frac{x}{1-2x}=x+2x^2+4x^3+8x^4+\cdots 2^{n-1}x^n+\cdots,\quad -\frac{1}{2}<x<\frac{1}{2}$$

$$3.\ f(x)=\frac{3}{1-x^3}=3+3x^3+3x^6+3x^9+\cdots 3x^{3n}+\cdots,\quad -1<x<1$$

Day 3

Derivatives and Integrals of Power Series

$$Find\ \frac{d}{dx}\sum_{n=0}^{\infty}3x^n$$

n=0	n=1	n=2	n=3	n

$$\sum_{n=0}^{\infty}3x^n = 3 \quad +3x \quad +3x^2 \quad +3x^3 \ldots +3x^n+\ldots$$

$$\frac{d}{dx}\sum_{n=0}^{\infty}3x^n = 0 \quad +3 \quad +2\cdot 3x \quad +3\cdot 3x^2 \ldots +3nx^{n-1}$$

$$\frac{d}{dx}\sum_{n=0}^{\infty}3x^n=\sum_{n=1}^{\infty}3nx^{n-1}$$

$$Find\ \frac{d}{dx}\sum_{n=0}^{\infty}4(2x)^n$$

Chain Rule

$$\frac{d}{dx}\sum_{n=0}^{\infty}4(2x)^n=\sum_{n=1}^{\infty}4n(2x)^{n-1}(2)$$

NOTE

$$=\sum_{n=1}^{\infty}8n(2x)^{n-1}$$

Keep in mind . . .

- The function and the series are essentially equivalent within the interval of convergence
  - Therefore, the RESTRICTED DOMAIN of the function will be the interval of convergence
- To integrate and take the derivative of functions written as a series
  - We use **term-by-term differentiation and integration of the series.** (Just like we do with polynomials)

Revisiting Calculus AB

- How are the following two functions related?

$$g(x)=\frac{1}{1-x} \qquad f(x)=\frac{1}{(1-x)^2}$$

Did you **MEMORIZE** this?

$$g(x) = \frac{1}{1-x}$$
$$= 1 + x + x^2 + \cdots + x^n + \cdots$$
$$= \sum_{n=0}^{\infty} x^n$$

Express $f(x) = \frac{1}{(1-x)^2}$ as a power series. State the radius of CV.

1) Pick a new function that is:

a. Related in a calculus way to given function

(derivative or antiderivative)

b. Can be easily written with power series representation

2) Convert to a Power Series

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

NOTE

3) Take the derivative or integrate to get the given function

$$g'(x) = \frac{1}{(1-x)^2} = f(x) = \sum_{n=1}^{\infty} nx^{n-1}$$

Radius of Convergence is the same as the picked series. R = 1

Write the first four terms without actually substituting "n" values

$$f(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Just so we keep this straight:

$$f(x) = g'(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

We differentiated term by term.

QUESTIONS??

Practice packet p. 2 #17

Step 1: Pick $g(x)$ such that $g'(x)$ or $\int g(x)dx$ is at least part of $f(x)$ AND with easily being able to write $g(x) = \sum$

Step 2: Write $g(x) = \sum$

Step 3: Manipulate $g(x) = \sum$ to get $f(x)$

Now moving onto integration . . .

An example is best . . . .

Find $\int \sum_{n=0}^{\infty} 3x^n dx$

$$\int \sum_{n=0}^{\infty} 3x^n dx = C + \sum_{n=0}^{\infty} \frac{3x^{n+1}}{n+1}$$

Find $\int \sum_{n=0}^{\infty} 5x^n dx$

$$\int \sum_{n=0}^{\infty} 5x^n dx = c + \sum_{n=0}^{\infty} \frac{5x^{n+1}}{n+1}$$

Another one…
Write out terms if you find that helpful

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

We do not increment the -1 exponent because both -1 and *n* are constants.

Now let's get crazy

Find the power series representation for

$$f(x) = \tan^{-1} x$$

First, we need a function that is either the derivative or antiderivative of $f(x) = \tan^{-1} x$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

Second, write this new function as a power series

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2\right)^n$$

Third, integrate both sides of:

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2\right)^n$$

$$\tan^{-1} x = \int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} \left(-x^2\right)^n dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\tan^{-1}(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Solve for the arbitrary constant.

Hint use a value of x that makes the series part disappear.

$$\tan^{-1}(0) = C + \sum_{n=0}^{\infty} (-1)^n \frac{(0)^{2n+1}}{2n+1}$$

$$0 = C + 0$$

$$C = 0$$

Final answer: $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

Why did we learn all of this?

To have the ability to integrate "icky" functions!

$$\int \tan^{-1}(x)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\,dx$$

One more for luck…

Find a power representation for $f(x) = \ln(1-x)$ and its radius of convergence.

First pick a calculus related function:

$$\frac{d}{dx}\left(\ln(1-x)\right) = -\frac{1}{1-x} = g(x)$$

$$\frac{d}{dx}\left(\ln(1-x)\right) = -\frac{1}{1-x} = g(x) = -\sum_{n=0}^{\infty} x^n$$

- Now integrate:

$$\ln(1-x) = -\int \frac{1}{1-x}\,dx \quad = -\int \sum_{n=0}^{\infty} x^n dx$$

$$= C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

Solving for C, let x =0

$$\ln(1-0) = C - \sum_{n=0}^{\infty} \frac{(0)^{n+1}}{n+1}$$

$$\ln(1) = C$$

$$0 = C$$

Final answer:

$$f(x) = \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$f(x) = \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

This series would allow us to calculate a transcendental function to as much accuracy as we like using only pencil and paper!

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad -1 < x < 1$$