BC Calculus

Unit 6 Day 2

Representations of Functions as Power Series

Warmup

Take out your HW.

1. SETUP the integral that could be used to find the length of a curve determined by

$$x(t) = 3t \text{ and } y(t) = 2t^2 \text{ from } t = 0 \text{ to } t = 9$$

2. The position of a particle moving in the xy-plane can be represented by the position vector

$$\left\langle e^{t^2}, e^{-t^3} \right\rangle$$

Find the velocity vector at time *t=3*.

Warmup--Answers

1.

$$\int_0^9 \sqrt{9+16t^2}\,dt$$

2.

$$\left\langle 6e^9, -27e^{-27} \right\rangle$$

HW Questions??

BC Calculus

Unit 6 Day 2

Representations of Functions as Power Series

Today:

Representing functions as a power series.

Why would we want to express a function as a sum of infinitely many terms?

Because it helps when integrating or taking the derivatives of tricky functions

(We will do this tomorrow!)

Let's explore . . .

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

This power series looks like a Geometric Series, with *a* = 1

r = x

A "variable" r

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

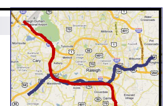
This series converges if . . . . $|x| < 1$

And will converge to… $\frac{a}{1-r}$ or $\frac{1}{1-x}$

$$\therefore\ 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

MEMORIZE

This means . . .



If $-1 < x < 1$ then

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

The series and the function are the same for this interval of x values.

If $x \le -1$ or $x \ge 1$

then the series and function diverge away from each other and there is no relationship

Practice Problems—Whiteboards or Notes (you decide)

Find the function of x represented by the following series and state the interval of convergence:

1. $\sum_{n=0}^{\infty} 2^n x^n$

Practice Problems—Whiteboards or Notes (you decide)

Find the function of x represented by the following series and state the interval of convergence:

2. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

Practice Problems—Whiteboards or Notes (you decide)

Find the function of x represented by the following series and state the interval of convergence:

3. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$

Now look at this in reverse…

- Given the function, $f(x) = \frac{1}{1-x}$

You may represent the function as a power series … $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad = \sum_{n=0}^{\infty} x^n$

- What would be the interval of convergence? (In other words, when is the above equation true?)

$$|x| < 1$$

Use your calculator to graph the function:

$$f(x) = \frac{1}{1-x}$$

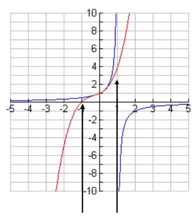
Now graph the following partial sums

$f(x) = \frac{1}{1-x}$

$S_2 = 1 + x$

$S_3 = 1 + x + x^2$

$S_4 = 1 + x + x^2 + x^3$



Notice: Interval of Convergence: (-1, 1)

Observations:

- *A function and series behave almost the same on the INTERVAL OF CONVERGENCE!!!*
- *Each partial sum is an approximation of f(x)*
- *The more terms that are included in the partial sum the better the approximation*

Express $f(x) = \frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Replace x by $-x^2$ in the following equation:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - \ldots$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - \ldots$$

$a = 1$

$r = -x^2$

$$= \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Geometric series, which converges when

$|-x^2| < 1 \Rightarrow -1 < x^2 < 1$

Interval of convergence: $(-1, 1)$

What about the endpoints?

$Interval\ of\ convergence: (-1, 1)$

The geometric sum formula, $\frac{a}{1-r}$, is not valid for r = 1.

Therefore, we cannot assume it works for these endpoints.

You can try testing the endpoints, but you should see divergence

Bottom line: we exclude the endpoints for these intervals

Let's look at the graph…

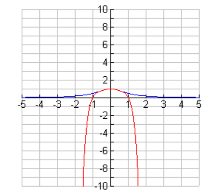
$$f(x) = \frac{1}{1+x^2}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \ldots$$

$$S_2 = 1 - x^2$$

$$S_3 = 1 - x^2 + x^4$$

$$S_4 = 1 - x^2 + x^4 - x^6$$



Notice: Interval of Convergence: (-1, 1)

Write $f(x) = \frac{1}{2+x}$ as a power series:

First manipulate the expression to get it in "standard" form:

$$\frac{1}{2+x} = \frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)} = \left[\frac{\frac{1}{2}}{1-\left(-\frac{x}{2}\right)}\right]$$

Write $f(x) = \frac{1}{2+x}$ as a power series:

$$\left[\frac{\frac{1}{2}}{1-\left(-\frac{x}{2}\right)}\right]$$

$$a = \frac{1}{2} \qquad r = -\frac{x}{2}$$

$$\frac{1}{2+x} = \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2} \frac{(-1)^n x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$

Write $f(x) = \frac{1}{2+x}$ as a power series:

$$\left[\frac{\frac{1}{2}}{1-\left(-\frac{x}{2}\right)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$

converges

$$\left|-\frac{x}{2}\right| < 1$$

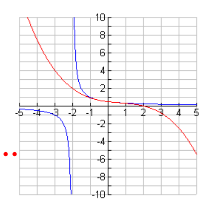
$$-2 < x < 2$$

Interval of Convergence is (-2,2)

Graphically…

$$f(x) = \frac{1}{2+x}$$

$$S_4 = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \ldots$$



Interval of Convergence: (-2, 2)

Find a power series representation of $f(x)=\dfrac{x^3}{2+x}$

$$\frac{x^3}{2+x} = \frac{\frac{x^3}{2}}{1-\left(-\frac{x}{2}\right)}$$

$a=\frac{x^3}{2}$

$r=-\frac{x}{2}$

$$\sum_{n=0}^{\infty}\frac{x^3}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^3}{2}\frac{(-1)^n x^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{2^{n+1}} = \sum_{n=3}^{\infty}\frac{(-1)^{n-3}}{2^{n-2}}x^n$$

Converges when $|r|<1$

$$\left|-\frac{x}{2}\right|<1$$

Interval of Convergence is (-2,2)

A second method

$$\frac{x^3}{2+x} = x^3\left(\frac{1}{2+x}\right)$$

We already determined that $\dfrac{1}{2+x}=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$

$$\text{So } x^3\left(\frac{1}{2+x}\right) = x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^3x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}$$

Rules for simplification

- Pull out (-1) if alternating series
- Combine common bases into single exponential
- Separate coefficient part from variable
- Adjust exponents and series bound to get x^n
- Adjust (-1) exponent to get *n-1, n, or n+1*

$$\sum_{n=0}^{\infty}\frac{x^3}{2}\left(-\frac{x}{2}\right)^n \longrightarrow = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3} \longrightarrow = \sum_{n=3}^{\infty}\frac{(-1)^{n-3}}{2^{n-2}}x^n$$

$$= \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{2^{n-2}}x^n$$

How do you know if your summation is correct?

$$1-x^2+x^4-\cdots \overset{?}{=} \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Expand out the first few terms of your series and verify you get the same list of terms.

$n=0$	$n=1$	$n=2$
$(-1)^0 x^{2(0)}$	$(-1)^1 x^{2(1)}$	$(-1)^2 x^{2(2)}$
$=1\cdot 1$	$=-1\cdot x^2$	$=1\cdot x^4$
$=1$	$=-x^2$	$=x^4$

Whiteboard Practice:

Find a power series to represent the given function and identify the interval of convergence

1. $f(x)=\dfrac{1}{1+3x}$

2. $f(x)=\dfrac{x}{1-2x}$

3. $f(x)=\dfrac{3}{1-x^3}$