Calculus BC Unit 5

Day 1 Agenda:
Notes on Factorials

Announcement

- HW needs to be completed DAILY for EVERYBODY
- Don't get behind! This unit can get difficult quick if you don't practice. Do your homework NIGHTLY.
- Remember you are essentially taking second semester college level Calculus. The material is rigorous.

Notes on Factorials

Important for later in this unit

Factorial

Informal Definition:
$n! =$ product of all integers from n to 1

Formal Definition of Factorial:

$$n! = \begin{cases} 1 & n = 0, n = 1 \\ n \times (n-1)! & n > 1 \end{cases}$$

So, a special case is 0!=1

Examples

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

$10! = 10\cdot 9\cdot 8\cdot\cdots\cdot 3\cdot 2\cdot 1 = 3,628,800$

$20! = 20\cdot 19\cdot 18\cdot\cdots\cdot 2\cdot 1 = 2.432\times 10^{18}$

Factorial on the Calculator

Enter the number first, then type

MATH → PRB → Option 4

$30! = 2.652\times 10^{32}$

$40! = 8.159\times 10^{47}$

$50! = 3.041\times 10^{64}$

Up for a race?

Who can determine the largest integer the calculator can find the factorial for?

We know it can do 50!

$69! = 1.711\times 10^{98}$

$70! = \text{Error}$

Brainstorm: Evaluate the limits

$$\lim_{n\to\infty} \frac{e^n}{n!} =$$

$$\lim_{n\to\infty} \frac{n!}{e^n} =$$

Which is growing faster?

$$\lim_{n\to\infty} e^n \quad or \quad \lim_{n\to\infty} n!$$

$$e^n = e\cdot e\cdots e\cdot e\cdots e\cdot e\cdots$$

$$n! = 1\cdot 2\cdot 3\cdots 10\cdots 100\cdots 1000\cdots$$

The factors of e^n are all e.

The factors of $n!$ are increasing without bound.

$$\lim_{n\to\infty} n! > \lim_{n\to\infty} e^n$$

Conclusion

$$\lim_{n\to\infty} \frac{e^n}{n!} = 0$$

$$\lim_{n\to\infty} \frac{n!}{e^n} = \infty$$

Expanding Factorials

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$$

$$n! = n(n-1)(n-2)\cdots\cdot 3\cdot 2\cdot 1$$

$$(n-1)! = (n-1)(n-2)\cdots\cdot 3\cdot 2\cdot 1$$

$$(n-2)! = (n-2)\cdots\cdot 3\cdot 2\cdot 1$$

$$(2n+1)! = (2n+1)(2n)(2n-1)\cdots\cdot 3\cdot 2\cdot 1$$

More examples

$$7! = (7)6!$$

$$7! = (7\cdot 6)5!$$

$$7! = (7\cdot 6\cdot 5)4!$$

$$n! = (n)(n-1)!$$

$$n! = (n)(n-1)(n-2)!$$

$$n! = n(n-1)(n-2)(n-3)!$$

Simplifying Factorials

$$\frac{5!}{7!}=\frac{5!}{(7\cdot 6)5!}=\frac{1}{7\cdot 6}$$

$$\frac{8!}{11!}=\frac{8!}{(11\cdot 10\cdot 9)8!}=\frac{1}{11\cdot 10\cdot 9}$$

$$\frac{59!}{57!}=\frac{(59\cdot 58)57!}{57!}=\frac{58\cdot 59}{1}$$

Expand the larger factorial until you have the same factorial in the numerator and denominator.

Then the factorials will divide out.

Simplifying factorials

$$\frac{(n-2)!}{n!}=$$

Before you start think about which factorial is bigger.

$$\frac{(n-2)!}{n(n-1)(n-2)!}=\frac{1}{(n-1)n}$$

A trickier example

$$\frac{(2n-1)!}{(2n+1)!}=$$

Before you start think about which factorial is bigger.

$$\frac{(2n-1)!}{(2n+1)(2n)(2n-1)!}$$

$$=\frac{1}{(2n)(2n+1)}$$

Why all the fuss about factorials?

$$f(x)=x^n$$

$$f'(x)=nx^{n-1}$$

$$f''(x)=n(n-1)x^{n-2}$$

$$f'''(x)=n(n-1)(n-2)x^{n-3}$$

$$f^n(x)=n(n-1)(n-2)\cdots 2\cdot 1=n!$$

Try It!

$f(x) = x^4$ Find $f^4(x)$ (the 4th derivative)

$$f' = 4x^3$$
$$f'' = 4 \cdot 3x^2$$
$$f''' = 4 \cdot 3 \cdot 2x^1$$
$$f^4 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$$

Again

$f(x) = x^{100}$

$f^{100}(x) = 100!$

And lastly

$f(x) = 4x^{50}$

$f^{50}(x) = 4(50!)$

Review/Correct of Unit 4 Test

UNUSED

$$n! = \begin{cases} 1 & n = 0, n = 1 \\ n \times (n-1)! & n > 1 \end{cases}$$

Using the formal definition

$$\begin{aligned} 4! &= (3!) \cdot 4 \\ &= (2! \cdot 3) \cdot 4 \\ &= (1! \cdot 2) \cdot 3 \cdot 4 \\ &= 1 \cdot 2 \cdot 3 \cdot 4 \end{aligned}$$

Conclusion: $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$