Day 3 – Asymptotes and End Behavior

Definition: An asymptote is an invisible boundary line in which the curve of a function will get very close to horizontally/vertically, but will not actually touch.

HOY = Horizontal Asy. have a slope of zero! $y = \#$

VERNOX = Vertical Asy. have NO slope! $x = \#$

When finding ***HORIZONTAL*** asymptotes, we need to compare the degree of the polynomials from the numerator and the denominator. (The degree is the HIGHEST power.)

top' bottom'

3 Cases: **BOBO/BOTNO/EATS DC**

"Bigger on Bottom" → zero

1) TOP degree (<) BOTTOM degree Example: $\frac{1}{x^2-1}$ therefore the H.A. = $y=0$

"Bigger on Top" → NOne

2) TOP degree (>) BOTTOM degree Example: $\frac{x^2+1}{x+3}$ therefore the H.A. = $y =$ Nothing

"Exponents Are The Same"

↳ Divide coefficients

3) TOP degree (=) BOTTOM degree Example: $\frac{1x^2+1}{3x^2+1}$ therefore the H.A. = $y = 1/3$

★ Make sure exponents are in descending order! ★

Examples: Find the Horizontal Asymptotes (if they exist) of the following functions.

$f(x) = \frac{4x^1+5}{4x^2-9}$ d=1, d=2 BOBO! $y=0$

$f(x) = \frac{3x^1+1}{1x^2-2}$ d=1 EATS D.C.! $y=3$

$f(x) = \frac{2x^1-1}{x^2+4}$ d=1, d=2 BOBO! $y=0$

$f(x) = \frac{x^3+2x+1}{x^2-x-12}$ d=3, d=2 BOTNO! NO H.A.

Steps for finding ***VERTICAL*** asymptotes:

1. Factor completely!
2. Divide out common factors
3. Set denominator = 0 & solve (!!)

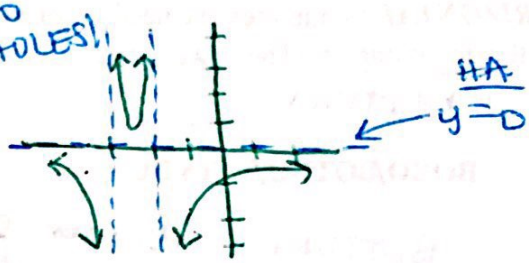
locations of HOLES

Examples: Find the Vertical Asymptotes (if they exist) of the following functions.

$f(x) = \dfrac{x-1}{x^2+5x+6} = \dfrac{(x-1)}{(x+2)(x+3)}$ NO HOLES!

$x+2=0$ $\quad x+3=0$

$x=-2$ $\quad x=-3$




V.A. $x=-2$ & $x=-3$

$f(x) = \dfrac{x^2+3x-10}{x^2-4} = \dfrac{(x+5)(x-2)}{(x+2)(x-2)}$ ← Hole @ $x=2$, $(2, 7/4)$

$\dfrac{(2)+5}{(2)+2} = \dfrac{7}{4}$

$x+2=0$

$x=-2$

HA: $y=1$

V.A. $x=-2$

$f(x) = \dfrac{4}{x^2+1}$

NOT factorable!

$x^2+1=0$

$-1 \quad -1$

$x^2 = -1$

$x = \pm i$ ← We don't have imaginary asymptotes!

V.A. none (!!)

YOU TRY!! Find the Vertical Asymptotes:

$f(x) = \frac{4x+5}{4x^2-9} = \frac{4x+5}{(2x+3)(2x-3)}$ $x = 3/2, \quad x = -3/2$

$f(x) = \frac{3x+1}{x-2}$ not factorable $x=2$

$f(x) = \frac{2x-1}{x^2+4}$ not factorable no V.A.

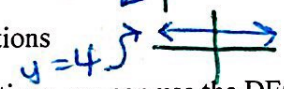
$f(x) = \frac{x^3+2x+1}{x^2-x-12} = \frac{x^3+2x+1}{(x+3)(x-4)}$ $x=-3, \; x=4$

End Behavior

The end behavior of a function describes the behavior of the graph as x approaches positive infinity or negative infinity. (How does the graph end on the left side and on the right side?)

- Left End: $\lim_{x\to-\infty} f(x) = ?$
- Right End: $\lim_{x\to\infty} f(x) = ?$

Possible Answers:

1. Positive/Negative Infinity  $\lim_{x\to-\infty} f(x) = \infty$ $\lim_{x\to\infty} f(x) = -\infty$
2. Horizontal Asymptotes   $\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to\infty} f(x) = 1$
3. Constant Functions   $\lim_{x\to\pm\infty} f(x) = 4$

**For polynomial functions, we can use the DEGREE and the LEADING COEFFICIENT to determine the End Behavior.

| DEGREE | LEADING COEFFICIENT | END BEHAVIOR | EXAMPLE |
|---|---|---|---|
| Even | Positive | $\lim_{x\to\infty} f(x) = \infty$ $\lim_{x\to-\infty} f(x) = \infty$ | $y = x^6 - 17$ |
| Even | Negative | $\lim_{x\to\infty} f(x) = -\infty$ $\lim_{x\to-\infty} f(x) = -\infty$ | $y = -x^6 - 17$ |
| Odd | Positive | $\lim_{x\to\infty} f(x) = \infty$ $\lim_{x\to-\infty} f(x) = -\infty$ | $y = x^3 + 2$ |
| Odd | Negative | $\lim_{x\to\infty} f(x) = -\infty$ $\lim_{x\to-\infty} f(x) = \infty$ | $y = -x^3 + 2$ |

same!

opp!

Examples:

$f(x) = x^2 + 3$ $\lim_{x\to\infty} f(x) = \infty$ $\lim_{x\to-\infty} f(x) = \infty$

$f(x) = \frac{3x^2}{x^2+1}$ HA: $y=3$ $\lim_{x\to\pm\infty} f(x) = 3$

$f(x) = \frac{2x-1}{x^2+4}$ HA: $y=0$ $\lim_{x\to\pm\infty} f(x) = 0$

Honors Pre-Calculus

Day 3 Homework

**For each function: 1) complete the table, 2) identify all asymptotes, and 3) describe the end behavior.**

1) $f(x) = \left(\frac{x+1}{2x-5}\right)$

$\frac{3}{2} = 1.5$

| $x$ | $y = f(x)$ |
|---|---|
| 500 | 1.5025 |
| 1,000 | 1.5013 |
| 10,000 | 1.50004 |
| 100,000 | 1.5 |
| 1,000,000 | 1.5 |

| $x$ | $y = f(x)$ |
|---|---|
| -500 | 1.4975 |
| -1,000 | 1.4958 |
| -10,000 | 1.49916 |
| -100,000 | 1.5 |
| -1,000,000 | 1.5 |

Horizontal Asymptote: $y = 1.5$ or $3/2$ Vertical Asymptote(s): $x = 5/2$ or $2.5$

Describe the end behavior using limit notation: $\lim_{x \to \pm\infty} f(x) = 1.5$

2) $f(x) = \frac{x^2+7}{5x-8}$

BOTH D

| $x$ | $y = f(x)$ |
|---|---|
| 500 | 100.32 |
| 1,000 | 200.32 |
| 10,000 | 2000.3 |
| 100,000 | 20000 |
| 1,000,000 | 200000 |

| $x$ | $y = f(x)$ |
|---|---|
| -500 | -99.68 |
| -1,000 | -199.7 |
| -10,000 | -2000 |
| -100,000 | -20000 |
| -1,000,000 | -199999.68 |

Horizontal Asymptote: none Vertical Asymptote(s): $x = 8/5$

Describe the end behavior using limit notation: $\lim_{x \to \infty} f(x) = \infty$ $\lim_{x \to -\infty} f(x) = -\infty$

3) $f(x) = \frac{3}{x}$

BOBD

| $x$ | $y = f(x)$ |
|---|---|
| 500 | 0.006 |
| 1,000 | 0.003 |
| 10,000 | 0.0003 |
| 100,000 | 0.00003 |
| 1,000,000 | 0.000003 |

| $x$ | $y = f(x)$ |
|---|---|
| -500 | -0.006 |
| -1,000 | -0.003 |
| -10,000 | -0.0003 |
| -100,000 | -0.00003 |
| -1,000,000 | -0.000003 |

Horizontal Asymptote: $y = 0$ Vertical Asymptote(s): $x = 0$

Describe the end behavior using limit notation: $\lim_{x \to \pm\infty} f(x) = 0$

**4) For each, identify all asymptotes and describe the end behavior using limit notation.**

a) $f(x) = -x^3 - 2$ NO Asy.

$\lim_{x \to -\infty} f(x) = \infty$

$\lim_{x \to \infty} f(x) = -\infty$

b) $f(x) = xe^{-x} = \frac{x}{e^x}$ BOBD

$y = 0$

$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = -\infty$

c) $f(x) = |3 - x|$

NO Asy.

$\lim_{x \to \pm\infty} f(x) = \infty$

d) $f(x) = \frac{-x^2}{x-1}$ $x = 1$

$\lim_{x \to 1} f(x) = -\infty$

e) $f(x) = \frac{4x-1}{x+3}$ $y = 4$ $x = -3$

$\lim_{x \to \pm\infty} f(x) = 4$

f) $f(x) = \frac{4x}{x^2+1}$ BOBD

$y = 0$

$\lim_{x \to \pm\infty} f(x) = 0$